

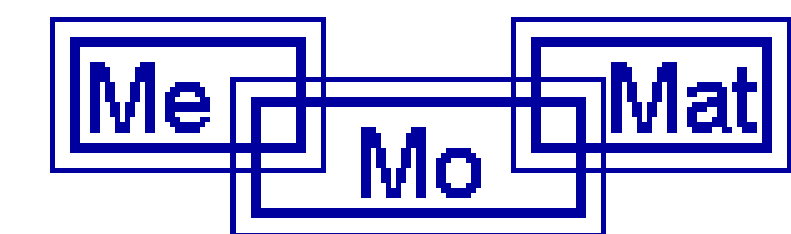
Integration of heterogeneous computational resources in EGEE infrastructure: a live demo (*an user case: POLY-SPAN*)

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POLY-SPAN a project exploring the problem of the span of polynomials with integer coefficients

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Project Motivation

Introduction

Polynomials with integer coefficients represent an argument of remarkable interest, studied by eminent scientists like Stieltjes, Kronecker, Chebyshev, Hermite, Schur, Pólya. Beyond their intrinsic interest in various fields of algebra and analysis, these polynomials have a remarkable importance for the applications, such as probability, physics and engineering.

A problem of great interest is to count the number of real zeros of a polynomial with integer coefficients in an assigned dominion.

By the **span of a polynomial $f(x)$** , we mean the **difference between the largest and smallest root of an algebraic equation having only real roots**. We consider monic irreducible equations with integer coefficients, so that the roots are a set of conjugate algebraic integers. Two equations are considered equivalent if the roots of one can be obtained from the roots of the other by adding an integer, changing signs, or both.

The problem

It is known that span greater than 4 must contain infinitely sets of conjugate algebraic integers, whereas an interval of length less than 4 can contain only a finite number of such sets. The problem remains open for intervals of length 4, except when the end points are integers.

In this case Kronecker determined the infinite family of polynomials of such type and showed that there are no other algebraic integers which lie with their conjugates in $[-2, 2]$. So there are infinitely many inequivalent algebraic equations with span less than 4, but for example, only a finite number with span less than 3.9.

Thus it appears that algebraic equations with span less than 4 are of particular interest. There are, of course, only a finite number of inequivalent equations of a given degree.

A basic work on such argument is due to Robinson who classified them, up to the degree 6 and was able to study them up to the degree 8 only partially, because of the computational complexity of the problem.

This project is an ideal continuation of Robinson's work, with the tool of modern computers and with a refined procedure.

Method of computation

Taking advantage of reasonable restrictive conditions, based on statistics analysis, we obtain a list of representative equations, one for each class of equivalent equations, with degrees from 2 to 16.

The representative equation must always be chosen in such a way that the average of the roots lies in $[0, \frac{1}{2}]$. If the average of the roots is 0 or $\frac{1}{2}$, then an ambiguity remains and the equation whose roots are in the shortest possible interval centred at $\frac{1}{4}$ is chosen. Chebyshev polynomials are used in the polynomial approximation.

For each polynomial degree the problem must be solved for a large number of sets of the polynomial integer coefficients. From a numerical point of view the solution is a typical multicase problem, well adapted for the GRID environment.

Implementation on the GRID

The software tool, selected by the project is **PariGP** (<http://pari.math.u-bordeaux.fr/>) one of the most used algebraic software oriented to calculus in number theory.

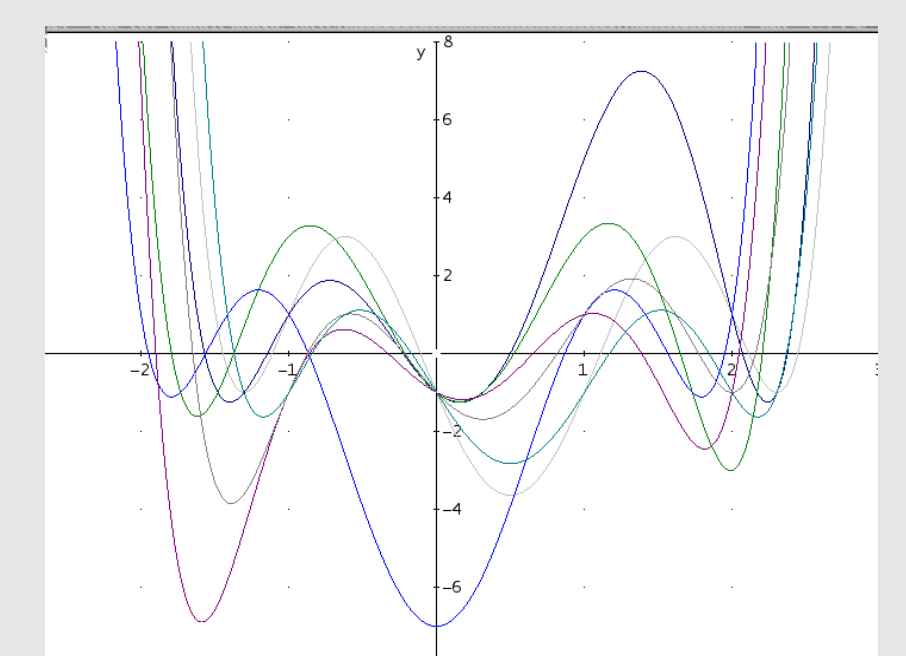
This software is under GPL licence and is a multiplatform code available for most of the existing OS/Platforms. It consists of an interface and a core code, called gp.

The gp code has been compiled for linux x86, AIX, altix (Linux IA64) and Mac OSX. We have installed the binary files in a shared geographically distributed filesystem (Open AFS). A new tag for gLite information system has been added [Parisgp] and the jobs are run by specifying the requirement "Parigp" in their jdl file.

In the production runs (~5k jobs at the moment) mostly the Linux x86 and AIX platforms have been used but tests have been performed also on Mac OSX and Altix systems.

The results

The figure on the right shows as an example the polynomials of degree 6 with span less than 4.

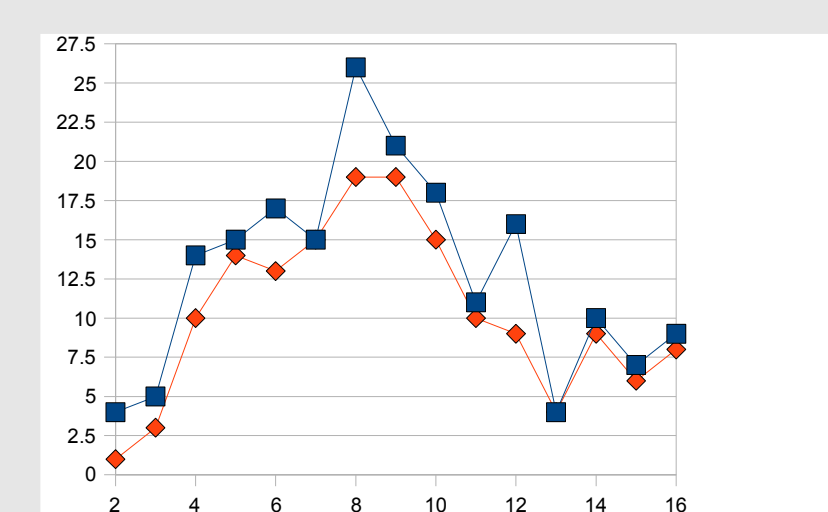
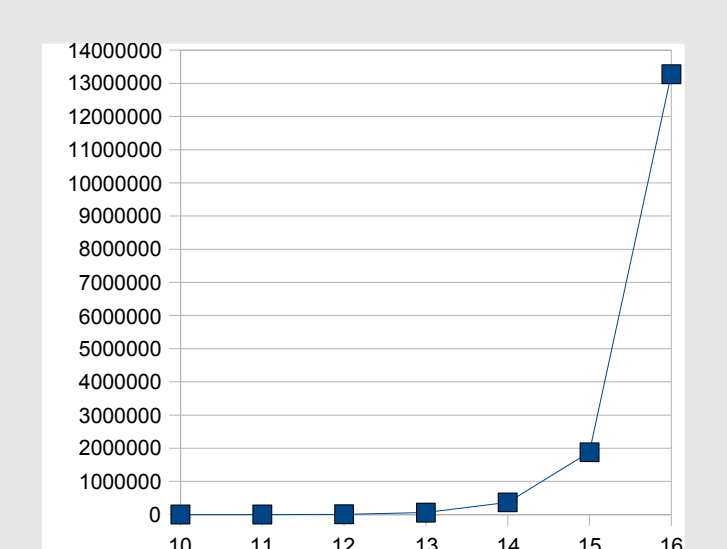


The table below illustrates the computational complexity of the problem

n	CPU time (sec)	Ratio (n+1)/n	N. CPU	Time/CPU (sec)	max single CPU time
10	224	5.91	33	6.79	9.34
11	1324	6.57	36	36.78	47.82
12	8705	6.86	48	181.35	263.81
13	59749	6.17	50	1194.98	1768.02
14	368793	5.1	63	5853.86	8516.26
15	1879041	7.06	1005	1869.69	4585.21
16	13274121	—	1316	10086.72	23762.68

The n index is the polynomial degree, followed by the total CPU time used. The third index represents the ratio between Cpu time in n+1 and n degree.

In the figure on the right the exponential trend of complexity versus the polynomial degree is shown.



The plot on the left shows the number of polynomials that do not satisfy the Kronecker condition, versus the polynomial degree.

Preliminary conclusions

The current conclusion of the project activity confirms the Robinson conjecture.

As a new result, we have observed that the number of the polynomial that do not satisfy the Kronecker conditions, seems drastically to decrease with increasing polynomial degree as shown in the last figure.

The final analysis is still in progress but it seems reasonable to conjecture that there are a finite number of such polynomial. Another interesting observation is the apparent strong correlation between the smallness of the distance between the nearest roots of a polynomial with its reducibility.