PLUS Seminar, 23 November 2021

Resonant axisymmetric modes in Tokamak plasmas

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OUTLINE

INTRO – X-point resonance, vertical displacements, feedback stabilization.

PART I – The math of X-point singularity, and (perhaps) a surprising result.

PART II – A new fast ion instability?

CONCLUDING REMARKS – A new line of research.

INTRODUCTION

Any unexplored concepts in tokamak plasma theory out there?

X-point

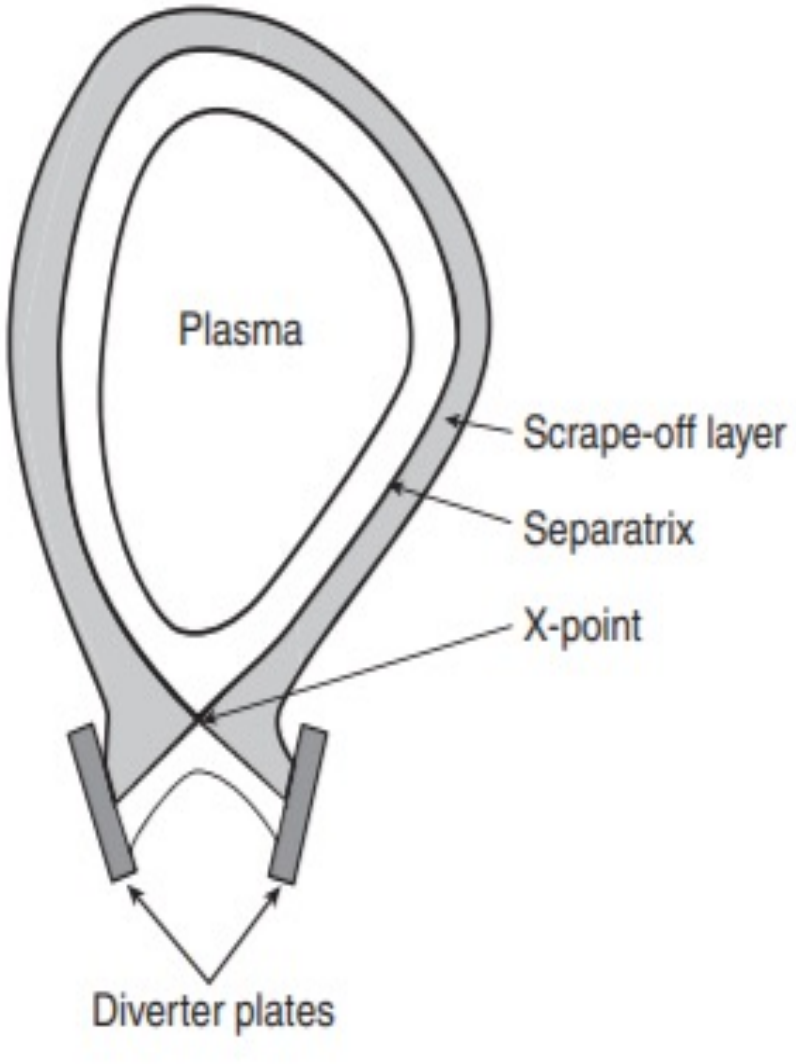
$\mathbf{B}\cdot\nabla\xi = i(n/R)B_T\,\xi + i\,B_P(\psi, \vartheta)\,\Sigma_m\,(m/r)\,\xi_m = 0$

Resonance condition at the X-point

satisfied for n=0 and any m

because $B_P(\psi, \vartheta)=0$ on the X-point.

The toroidal field line going through the magnetic X-point is resonant to n=0 perturbations!



Schematic diagram of a divertor (single null configuration)

Axisymmetric Modes

Axisymmetric (n=0) modes in shaped toroidal plasmas are global plasma modes, normally associated with VDEs.

Because of the X-point resonance, **current sheets localized along the magnetic separatrix are likely to form.** This may have an important impact on plasma edge activity.

Ref.: A. Yolbarsop, F. Porcelli, R. Fitzpatrick, Nuclear Fusion Letters 2021.

Furthermore, a stable **n=0 mode tends to acquire an Alfvénic oscillation frequency**. As a result, a **"resonant" interaction with energetic particles** can lead to a new type of fast ion instability.

Ref.: T. Barberis, F. Porcelli and A. Yolbarsop, submitted for publication (Oct. 2021)

But then, why nobody has worried about the n=0 resonance?

I am not sure…

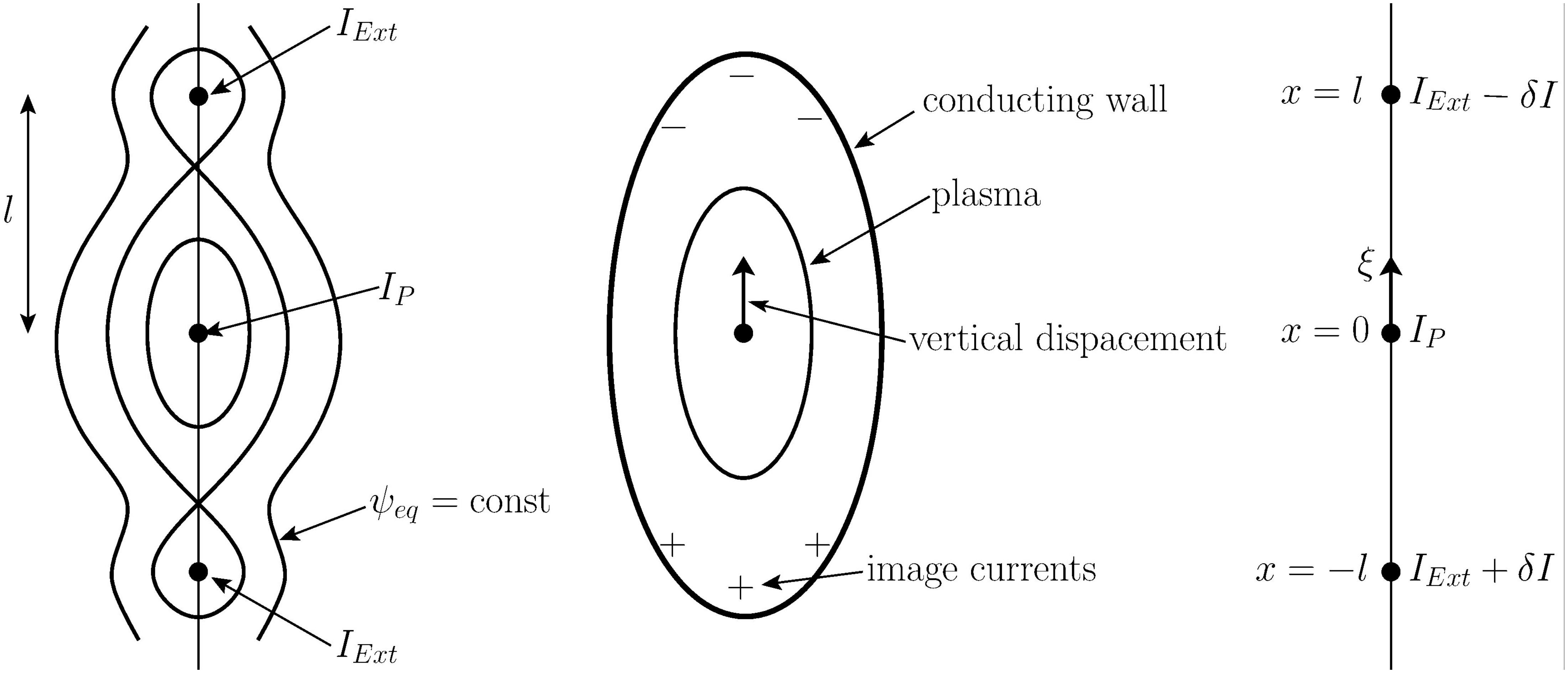
Perhaps… because these modes are taken care of by the feedback stabilization system?

Let us review the philosophy behind passive and active feedback stabilization of vertical plasma displacements.

A heuristic model:
three parallel current wires.

Lister et al, NF 1990;
Lazarus et al, NF 1990.

Three-current-wires model



Cubic dispersion relation

$$\mu\dddot{\xi} + \frac{1}{\tau_R}\ddot{\xi} + \omega_0^2\dot{\xi} - \frac{1}{\tau_R}\frac{\omega_0^2}{D-1} = 0$$

where

$$\omega_0^2 = (D-1)e_0\omega_{pA}^2; \qquad D = \frac{b^2+a^2}{(b-a)^2}\frac{b_w-a_w}{b_w}$$ (Laval et al, PF1974)

$D > 1$ for passive feedback stabilization

Three roots:

$$\omega = \pm\omega_0 - \frac{iD}{2(D-1)\tau_R}$$ stable, oscillatory

$$\omega = i\gamma = i\frac{1}{(D-1)\tau_R}$$ unstable (requires active feedback)

Note: only two oscillatory roots for $\tau_R \to \infty$: $\omega^2 = \omega_0^2$ (ideal wall limit)

Comments

- Only the weakly growing, zero frequency mode is considered for active feedback stabilization.
- **No X-point resonance effects.**
- **Hey! What about the weakly damped, high frequency solutions?**

PART I

The math of X-point singularity, and (perhaps) a surprising result.

For details, see:

IOP Publishing | International Atomic Energy Agency Nuclear Fusion
Nucl. Fusion 61 (2021) 114003 (5pp) https://doi.org/10.1088/1741-4326/ac27c5

Letter

Impact of magnetic X-points on the vertical stability of tokamak plasmas

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Reduced ideal-MHD model

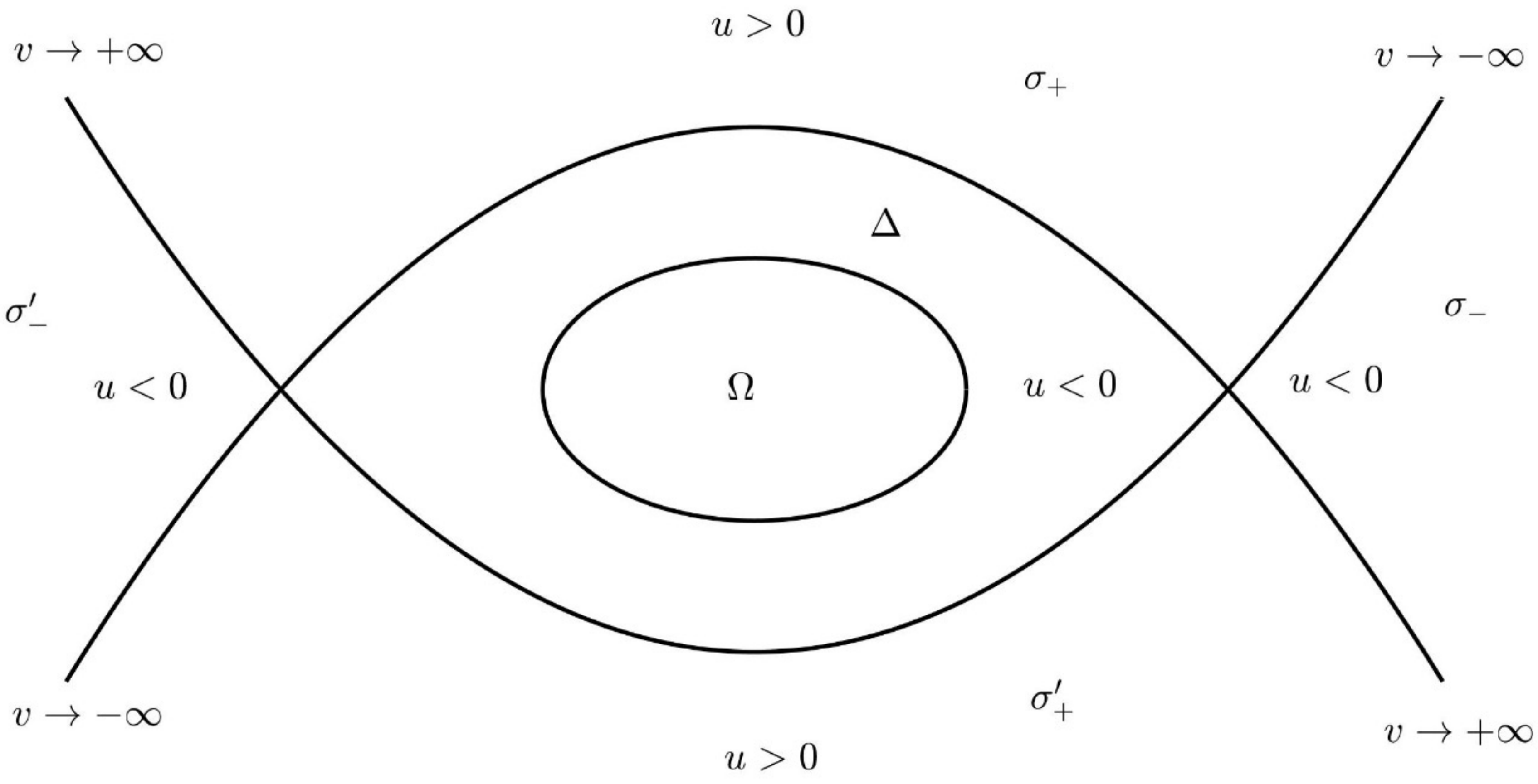
$$\mathbf{B} = \mathbf{e_z} \times \nabla\psi + B_z\mathbf{e_z}$$
$$\mathbf{v} = \mathbf{e_z} \times \nabla\varphi.$$

$$\frac{\partial\psi}{\partial t} + [\varphi, \psi] = 0$$

$$\frac{\partial}{\partial t}\nabla \cdot (\varrho\nabla\varphi) + [\varphi, U] = [\psi, J]$$

$$[\chi, \eta] = \mathbf{e_z} \cdot \nabla\chi \times \nabla\eta, \quad J = \nabla^2\psi, \quad U = \nabla^2\varphi.$$

Equilibrium magnetic structure



$$J_{eq} = J_0 H(\psi_b - \psi_{eq})$$
$$\varrho_{eq} = \varrho_0 H(\psi_X - \psi_{eq})$$

Plasma occupies Ω **and** Δ regions

Equilibrium

Uniform current density within an elliptical boundary

$$J_{eq} = J_0 \, H(\mu_b - \mu) = J_{eq}(\psi_{eq})$$

Elliptical coordinates

$$x = A \cosh \mu \cos \theta$$
$$y = A \sinh \mu \sin \theta$$
$$A = \sqrt{b^2 - a^2}$$
$$a = A \sinh \mu_b$$
$$b = A \cosh \mu_b$$

Equilibrium solution

$$[\psi_{eq}, J_{eq}] = 0 \quad \Longrightarrow \quad \nabla^2 \psi_{eq} = J_0 H(\mu_b - \mu)$$

Solution:

$$\psi_{eq}^{-}(x,y) = \frac{1}{2}\left(\frac{x^2}{b^2} + \frac{y^2}{a^2}\right)$$

Ω region

$$\psi_{eq}^{+}(\mu,\theta) = \frac{1}{2} + \alpha^2 \left\{\mu - \mu_b - \frac{e_0}{2}\sinh\left[2(\mu - \mu_b)\right]\cos(2\theta)\right\}$$

Δ and σ regions

$$\nabla^2\chi = h^{-2}(\partial^2\chi/\partial\mu^2 + \partial^2\chi/\partial\theta^2)$$
$$h = 1/|\nabla\mu| = 1/|\nabla\theta|$$

Refs: Porcelli&Yolbarsop, PoP 2019; Gajewski, PF 1972.

Perturbations:

The ideal-MHD constraint
at the magnetic X-point

and

the rigid-shift solution.

EXACT ANALYTIC SOLUTION

Rigid-vertical-shift stream function $\varphi = -\gamma\xi y$

Elliptical coordinates

$$x = A\cosh\mu\ \cos\theta$$

$$y = A\sinh\mu\sin\theta$$

$$A = (b^2 - a^2)^{1/2}$$

X- point coordinates $\mu = \mu_X = 2\mu_b$ and $\theta = \theta_X = 0, \pi$

Stream function in elliptical coordinates: $\varphi(\mu, \theta) = -\gamma A\sinh\mu\sin\theta$

Use flux-freezing condition to obtain perturbed flux

$$\gamma\,\tilde{\psi} + [\tilde{\varphi}, \psi_{eq}] = 0,$$

$$\tilde{\psi}_\Omega(\mu,\theta) = -\xi_0\frac{\cosh(\mu)}{\cosh\mu_b}\cos\theta.$$

$$\tilde{\psi}_\Delta(\mu,\theta) = \xi_0\frac{\sinh(\mu - 2\mu_b)}{\sinh\mu_b}\cos\theta.$$

Note:
single m=1
harmonic in
elliptical angle;

perturbed flux
vanishes at
X-points.

Vacuum solution (no wall)

$$\tilde{\psi}_{\sigma_\pm}(\mu,\theta) = \sum c_m \exp(-m\mu)\cos(m\theta).$$

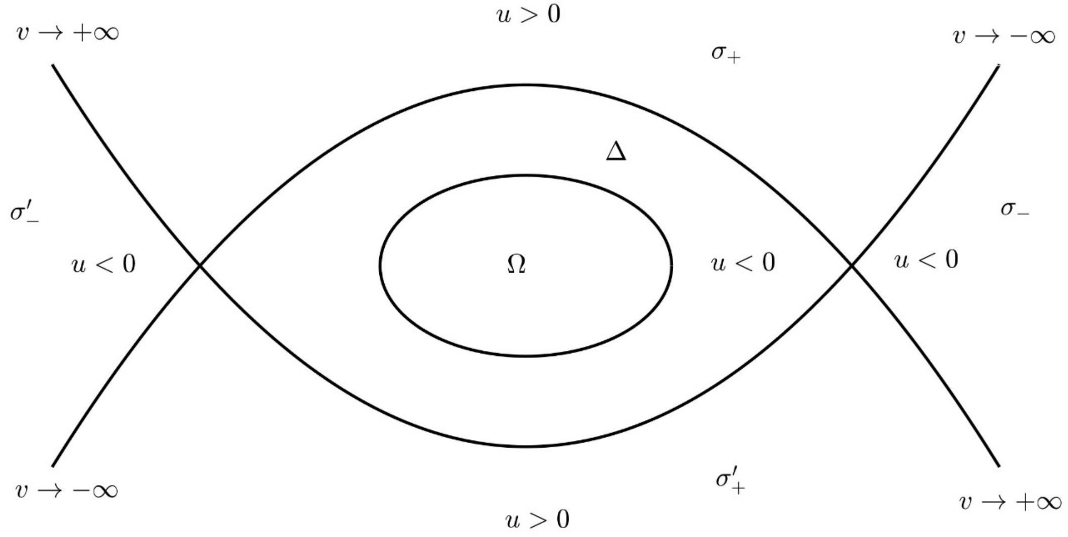
However, the separatrix is not a $\mu =$ const magnetic surface, and so analytic determination of coefficients c_m is an inherently intractable problem.

Analytic progress is possible if **flux coordinates** (u, v) are used:

$$u = \alpha^{-2}(\psi_{eq} - \psi_X) = \mu - 2\mu_b + \frac{e_0}{2}\{\sinh(2\mu_b) - \sinh[2(\mu - \mu_b)]\cos(2\theta)\}$$

$$v = \theta - \frac{e_0}{2}\cosh[2(\mu - \mu_b)]\sin(2\theta)$$

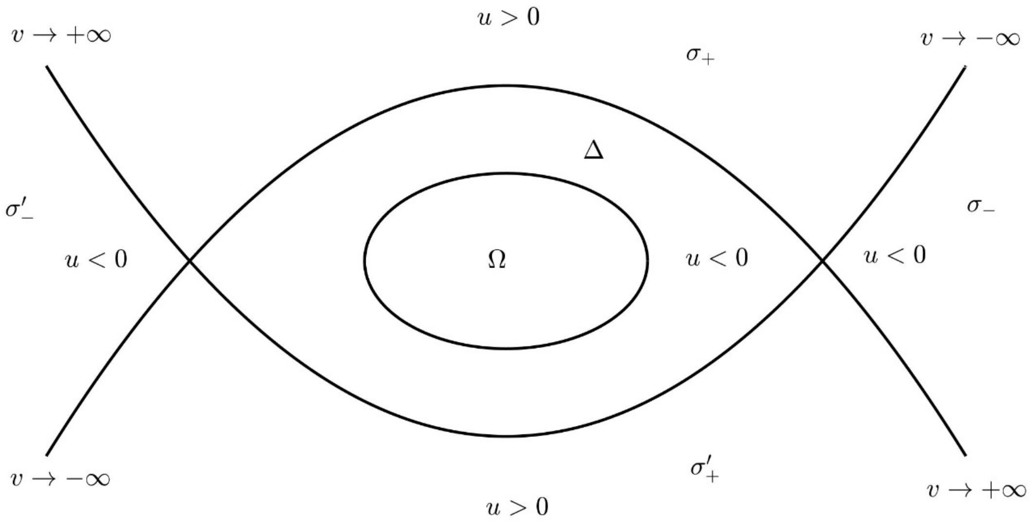
$$e_0 = \frac{b^2 - a^2}{b^2 + a^2},$$



Since $\nabla^2 \tilde{\psi}_\Delta = 0$

$$\tilde{\psi}_\Delta(u,v) = \xi_0 \sum_{m,odd}^{\infty} \left[\, \alpha_m \cosh(mu) + \gamma_m \sinh(mu) \,\right] \cos(mv).$$

$\tilde{\psi}_\Delta(u,v) = \tilde{\psi}_\Delta(\mu,\theta)$. We can use this to determine the α_m, γ_m coefficients.



$$\tilde{\psi}_\Delta(\mu,\theta) = \xi_0 \frac{\sinh(\mu - 2\mu_b)}{\sinh \mu_b} \cos\theta.$$

After straightforward algebra, we obtain $\alpha_m = a_m + b_m$, $\gamma_m = -a_m + b_m$

$$a_m = \frac{e^{mu_b}}{2m}\left[(\frac{b}{a}+1) J_{\frac{m-1}{2}}(\frac{me_0}{2}) + (\frac{b}{a}-1) J_{\frac{m+1}{2}}(\frac{me_0}{2}) \right]$$

$$b_m = \frac{e^{-mu_b}}{2m}\left[(\frac{b}{a}-1) J_{\frac{m-1}{2}}(\frac{me_0}{2}) + (\frac{b}{a}+1) J_{\frac{m+1}{2}}(\frac{me_0}{2}) \right]$$

Mathematics of X-point singularity: non-analiticity

Consider magnetic flux on separatrix

$$\psi_\Delta(0,v) = \sum_{m\ odd} \alpha_m \cos mv$$

Fourier spectrum al large m

$\alpha_m \sim p e_0^{1/2}/m^{3/2}$ → $\psi_\Delta(0,v)$ NOT an analytic function of v

Indeed:

$\psi_\Delta(0,v) \sim \sqrt{\frac{\pi}{2}} p e_0^{1/2} |v|^{1/2}$ as $v \to 0$

$p \approx (2e_0/\pi)^{1/2}$ for small e_0

Continuity along the separatrix determines the perturbed flux in the vacuum region. This involves a parameter q.

Along the separatrix:

$$\tilde{\psi}_{\sigma^+}(0,v) = \int_0^\infty dt\, \alpha(t) \sin\left[\left(\frac{\pi}{2} - v\right) t\right] = \tilde{\psi}_{even}(0,v) + \tilde{\psi}_{odd}(0,v)$$

even: q = p/2
odd: q = - p/2

$p \approx (2e_0/\pi)^{1/2}$ for small e_0

Current sheet along the separatrix:

$$\tilde{J}(u,v) \sim j_X(v)\delta(u)$$

where

$$j_X(v) \sim e_0^{3/2}\left(\frac{p}{2}+q\right)\xi_0|v|^{1/2} \quad \text{as} \quad v \to 0$$

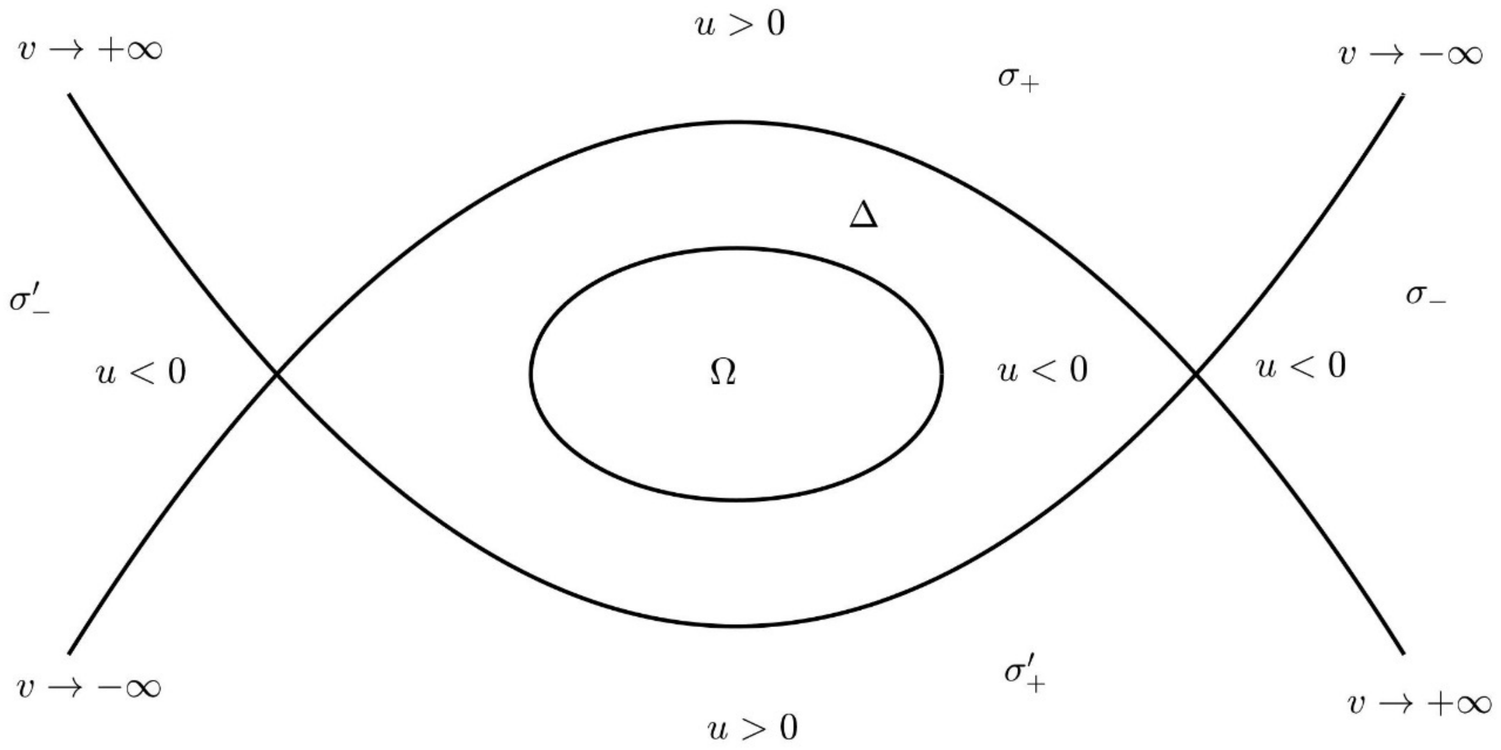
Growth rate: $$\gamma^2 = -\sqrt{\frac{\pi a}{2b}}\frac{1}{\alpha^2}\left(q+\frac{p}{2}\right)e_0^{3/2}\omega_A^2$$

Two classes of modes:

even: q=p/2 - stable oscillatory roots - current sheet on separatrix
odd: q=-p/2 - neutrally stable with $\gamma = 0$ - no current sheet

$p \approx (2e_0/\pi)^{1/2}$ for small e_0

Mathematics of X-point singularity:
v non-periodic in vacuum region



Conclusions

The mathematics of the X-point singularity is established!

n=0 current sheets form along the magnetic separatrix, with $\tilde{J} \sim |v|^{1/2}$ as the X-point (at $v = 0$) is approached.

The surprising result is that, because of these current sheets, the ideal-MHD vertical displacement is stable even in the absence of a passive feedback stabilization wall.

Conclusions Part I

The stabilization mechanism is a direct consequence of the ideal-MHD flux-freezing constraint on the X-points, which generates current sheets localized along the magnetic separatrix, exerting a force capable of pushing back the plasma in its vertical motion.

Analogies with the physics of current sheet formation from the nonlinear evolution of internal kink modes (*) and with the island coalescence problem (**) → the ideal-MHD constraint causes magnetic flux to pile up near the X-points, leading to perturbed localized currents and a stabilizing effect in the ideal-MHD limit.

(*) M. N. Rosenbluth, N. Y. Dagazian, and P. H. Rutherford, Phys. Fluids 16, 1894 (1973).

(**) J. M. Finn and P. K. Kaw, Phys. Fluids 20, 72 (1977).

Have these axisymmetric X-point current sheets been observed?

Perhaps yes…

JET experiments:

- J. Lingertat et al., J. Nucl. Mat. 241, 402 (1997);
- E. R. Solano et al., Nucl. Fusion 57, 022021 (2016).

Numerical simulations (M3D-C^1; NIMROD; JOREK):

- I. Krebs, F. G. Artola, C. R. Sovinec, S. C. Jardin, K. J. Bunkers, M. Hoelzl, and N. M. Ferraro, Phys. Plasmas 10, 930 (2020).

PART II

A new fast ion instability?

One branch of the n=0 dispersion relation corresponds to oscillatory modes with a discrete frequency,

$$\omega \approx \pm e_0^{1/2}\omega_{pA}$$

not interacting with the Alfvén continuum, weakly damped by wall and/or plasma resistivity.

We find that this mode can be destabilized by the resonance with fast ions on both trapped and passing orbits

Experimental evidence

Observation of n=0 modes

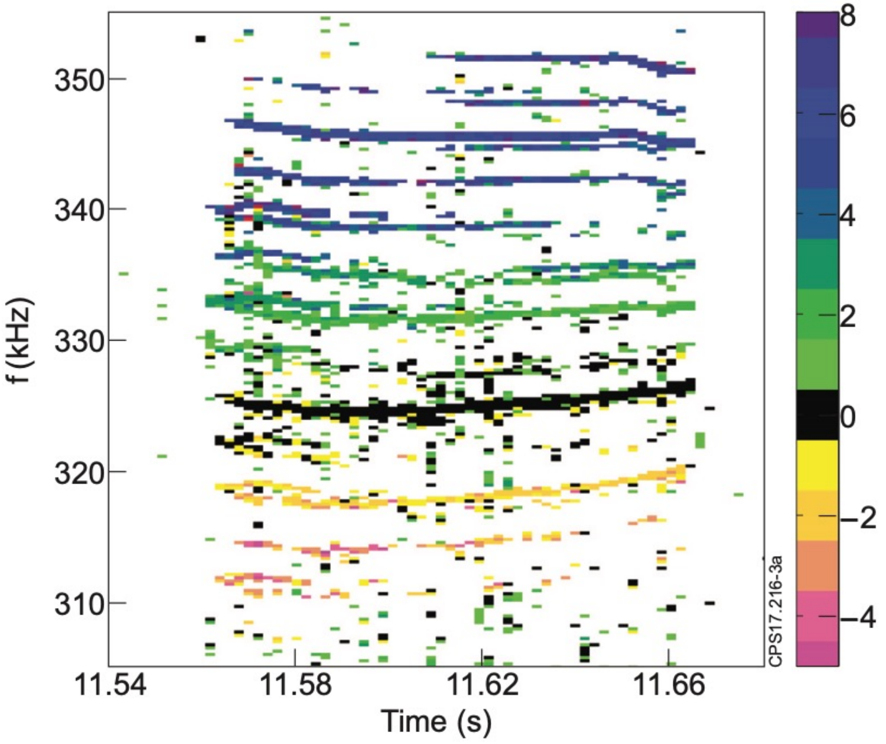


FIG. 3. Phase magnetic spectrogram showing the toroidal mode numbers of modes in the EAE frequency range excited in discharge #86775. Modes with toroidal mode numbers $-5 \leq n \leq 8$ were excited for a period of 0.1 s following a monster sawtooth crash at $t = 11.48$ s. $n = 0$ is shown in black.

H. J. C. Oliver et al. 2017 PoP

Observation of saturated n=0 modes in D-^3He plasmas at JET with NBI-ICRF.

$\partial f/\partial E > 0$ was concluded to be the source of free energy to destabilize this mode, which was identified as a GAE.

Linearized Kinetic-MHD model

$$\gamma\tilde{\psi} + [\tilde{\varphi}, \psi_{eq}] = 0$$

$$\gamma\nabla \cdot (\varrho_{eq}\nabla\tilde{\varphi}) = [\tilde{\psi}, J_{eq}] + [\psi_{eq}, \tilde{J}] - \boldsymbol{e_z} \cdot \nabla\times(\nabla \cdot \widetilde{\boldsymbol{P}}_{\boldsymbol{h}})$$

$$\widetilde{\boldsymbol{P}}_{\boldsymbol{h}} = \tilde{p}_{\perp h}\boldsymbol{I} + (\tilde{p}_{\parallel h} - \tilde{p}_{\perp h})\boldsymbol{e}_{\parallel}\, \boldsymbol{e}_{\parallel}$$

Dispersion relation

$$-\gamma^2 \int d^3x \rho \boldsymbol{\xi} \cdot \boldsymbol{\xi}^* = -\frac{1}{2}\int d^3x \boldsymbol{\xi}^* \cdot F(\boldsymbol{\xi}) + \frac{1}{2}\int d^3x \boldsymbol{\xi}^* \cdot \nabla \cdot \widetilde{\boldsymbol{P}}_{\boldsymbol{h}} = \delta W_{MHD} + \delta W_h$$

$\widetilde{\boldsymbol{P}}_{\boldsymbol{h}}$ is obtained in terms of moments of the perturbed fast ion distribution function.

In the limit $\beta_h \ll \beta_i$, $\delta W_h \ll \delta W_{MHD}$ and only the imaginary part of δW_h matters.

Perturbative approach

To zeroth order, fast particles are neglected → rigid-shift solution

To first order, hot particle corrections are considered → δW_h is evaluated using the zeroth order rigid-shift solution.

The dispersion relation takes on the form

$$\omega^2 = \omega_0^2 - 2i\omega_0\gamma_\eta + \mathrm{Im}(\delta\widehat{W}_h)$$

Let $\mathrm{Im}(\delta\widehat{W}_h) = i\omega_0^2\lambda_h + \mathcal{O}\left(\frac{\gamma^2}{\omega_0^2}\right)$, $\omega = \omega_0 + \delta\omega$, and $\delta\omega = \mathcal{O}(\gamma/\omega_0)$. Then:

$$\delta\omega = -i\gamma_\eta + i\frac{1}{2}\omega_0\lambda_h = i\gamma$$

Instability requires $\lambda_h > \frac{2\gamma_\eta}{\omega_0}$.

Fast particle δW_h

$$\delta W_h = -\frac{1}{2}\int d^3x d^3v\, \tilde{L}^* \tilde{f}_h$$

where $\tilde{L} \simeq -(mv_\parallel^2 + \mu_\perp B)\boldsymbol{\xi}\cdot\boldsymbol{\kappa}$ is the perturbed Lagrangian for fast particle motion. Now:

$$\delta W_h(\omega) = \delta W_1 + \delta W_2(\omega)$$

$$\delta W_2 = -\frac{2\pi^2 c}{Zem^2}\sum_\sigma \int dP_\varphi d\mathcal{E} d\mu_\perp \tau_t \left(\omega \frac{\partial f_h}{\partial \mathcal{E}} - n \cancel{\frac{\partial f_h}{\partial P_\varphi}}\right) \sum_{p=-\infty}^{+\infty} \frac{\left|\Upsilon_p\right|^2}{\omega + p\omega_t}$$

$$\Upsilon_p(\mathcal{E},\mu,P_\varphi) = \langle \tilde{L}\exp(ip\omega_t\tau)\rangle \quad \text{and} \quad \Upsilon_1 = i\frac{\epsilon^2}{q(r)^2}\mathcal{E}(2-\Lambda)\frac{\xi}{r}\langle \sin(\vartheta)\sin(\tau\omega_t)\rangle$$

Resonant Interaction

$$\Lambda = \frac{\mu B_{\varphi_0}}{\mathcal{E}}\,,\ 0 \le \Lambda \le 1 + \epsilon$$

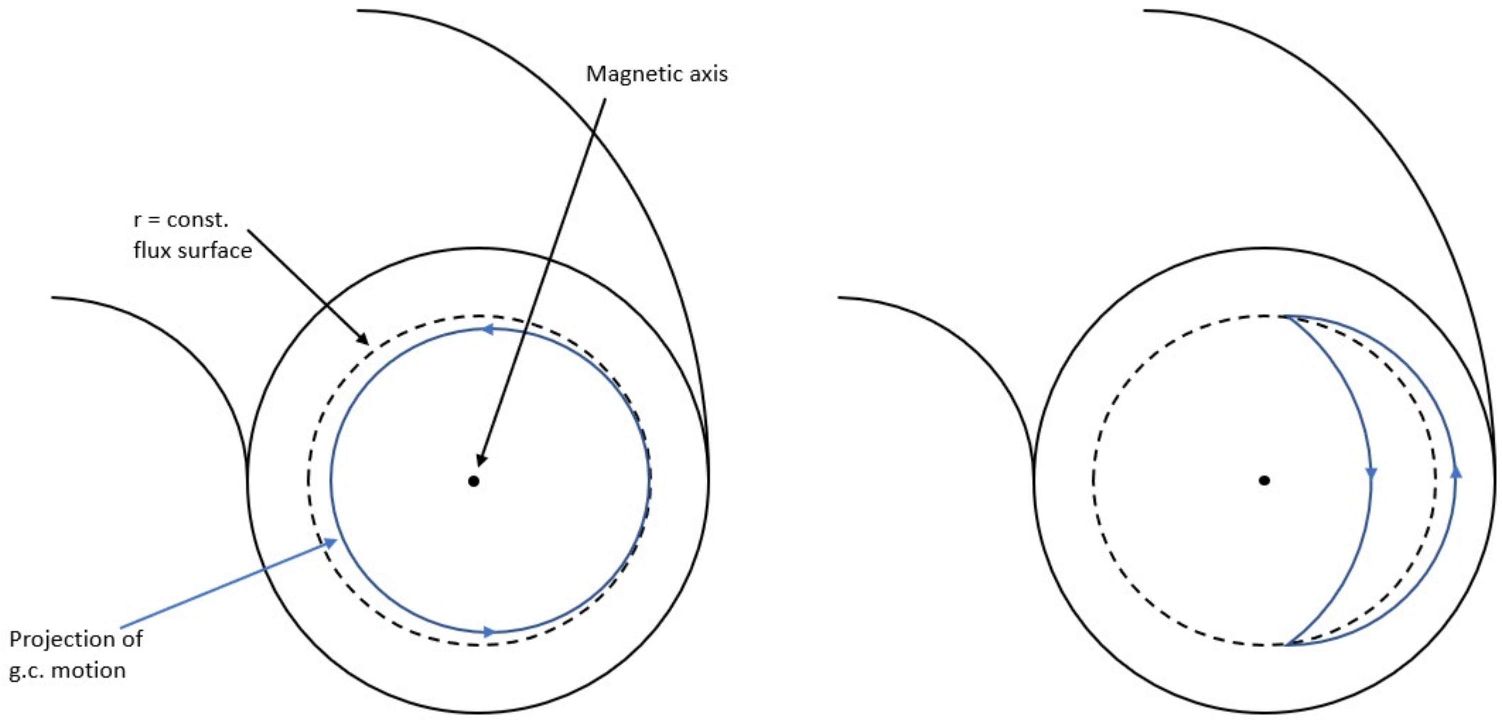
$$\tau_{t/b} = \oint d\tau$$

$$\omega_t = \frac{2\pi}{\tau_t} = \frac{v\kappa}{2R_0 q}(2\epsilon)^{1/2}\frac{\pi}{\mathcal{K}(1/\kappa^2)}$$

$$\omega_b = \frac{2\pi}{\tau_b} = \frac{v}{2\sqrt{2}R_0 q}(\epsilon)^{1/2}\frac{\pi}{\mathcal{K}(\kappa^2)}$$

$$\kappa^2 = \frac{1}{2} + \frac{1}{2\epsilon}(1 - \Lambda)$$

Fast ions trajectories



(a) $\Lambda \le 1 - \epsilon$

(b) $1 - \epsilon < \Lambda \le 1 + \epsilon$

$\partial f_h/\partial E > 0$ required for instability

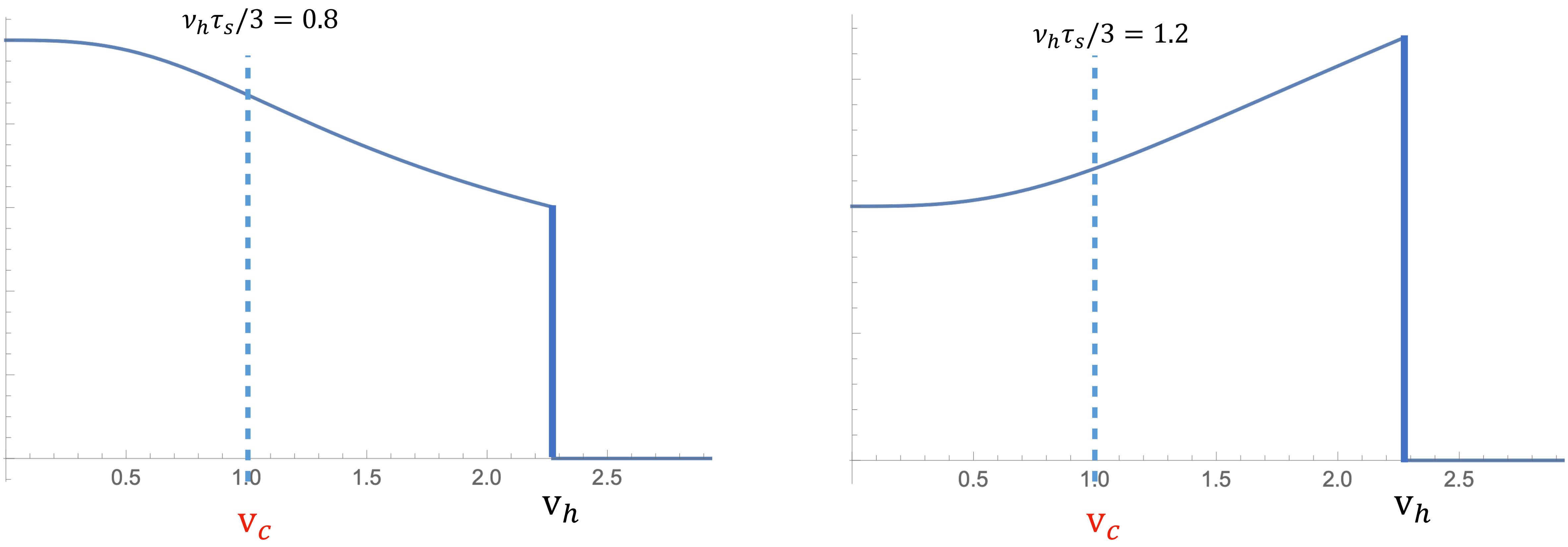
Fokker-Plank equation for fast particle distribution function:

$$\frac{\partial f_h}{\partial t} - \frac{1}{\tau_s \mathrm{v}^2}\frac{\partial}{\partial \mathrm{v}}[(\mathrm{v}^3 + \mathrm{v}_c^3)f_h] + \nu_h f_h = S_0\delta(\mathrm{v} - \mathrm{v}_h)$$

ν_h: fast particle loss frequency; τ_s: slowing down time

Positive slope obtained for $\frac{\nu_\alpha \tau_s}{3} > 1$.

Solution of F.-P. equation for different values of $\nu_h \tau_s/3 < 1$



Experimental evidence

M.A. Van Zeeland et al., PoP 2021:

Beam Modulation and Bump-on-tail Effects On Alfvén Eigenmode Stability in DIII-D

Insurgence of bump-on-tail fast ion distribution modulating NBI with periods 7-30 ms (reported: $\tau_s \sim 50\text{ms}$).

V.G. Kiptily et al., NF 2021

Self modulation of alpha particles production due to sawtooth oscillations in JET D-He^3 experiments.

Bump-on-tail distribution for energetic particles when the sawtooth period is relatively short, i.e., $\tau_{saw}/\tau_s < 1$.

Critical instability threshold – numerical estimates

$$\lambda_h > \frac{2\gamma_\eta}{\omega_0} \quad \rightarrow \quad \left(\frac{n_h}{n_i}\right)_{crit} = \frac{2\gamma_\eta}{\alpha\omega_0}$$

$$\lambda_h = \alpha(n_h/n_i); \qquad \alpha \sim 10^{-1} - 10^{-2}; \qquad \gamma_\eta/\omega_0 \sim 10^{-4}$$

$$\left(\frac{n_h}{n_i}\right)_{crit} \sim \ 10^{-3} \ - 10^{-2}$$

CONCLUDING REMARKS

Have we forgotten anything?

Check-list for n=0 developing theory

- ☑ Ideal-MHD
- ☐ Resistive-MHD
- ☐ Extended-MHD
- ☑ Hybrid Kinetic-MHD
- ☐ SOL Physics, field-line-tying, special boundary conditions
- ☐ other…

Not only n=0 vertical displacements

- Axisymmetric current sheet at the magnetic X-point and along the last closed flux surface may affect the stability of peeling modes
- The nonlinear evolution of peeling-ballooning modes is likely to produce a nonlinear n=0 component that resonate at the X-point
- ELM mitigation by vertical kicks
- ….

Thank you for your attention!