Linear and non Linear Theory of Accretion Disks

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June 28, 2019

1 Physical Model

- a steady and axialsymmetric thin disk configuration, with a magnetic field $\vec{B}$ having the following poloidal form

$$\vec{B} = -r^{-1}\partial_z\psi\hat{\mathbf{e}}_r + r^{-1}\partial_r\psi\hat{\mathbf{e}}_z\,, \tag{1}$$

- standard cylindrical coordinates $(r\,,\phi\,,z)$ ($\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$ being their versors)
- $\psi(r,z)$ is the magnetic flux function.
- The disk also possesses a purely azimuthal velocity field $\vec{v}$, id est

$$\vec{v} = \omega(\psi)r\hat{\mathbf{e}}_\phi\,, \tag{2}$$

- ω is a function of ψ (corotation theorem)
- We now split the magnetic flux function around a fiducial radius $r = r_0$, as follows

$$\psi = \psi_0(r_0) + \psi_1(r_0, r - r_0, z)\,, \tag{3}$$

where ψ_0 is the vacuum contribution of the central object around which the disk develops, r_0 is a fiducial radius

- the magnetic field is expressed as $\vec{B} = \vec{B}_0 + \vec{B}_1$ with $|\vec{B}_1 << \vec{B}_0$
-

$$\omega(\psi) \simeq \omega_0(\psi_0) + (d\omega/d\psi)_{\psi=\psi_0}\,\psi_1\,. \tag{4}$$

$$\omega_0(\psi_0) = \omega_K\,, \tag{5}$$

$$\partial_r^2\psi_1 + \partial_z^2\psi_1 = -k_0^2(1 - z^2/H^2)\psi_1\,, \tag{6}$$

- ω_K denotes the Keplerian disk angular velocity, H is the half-depth of the disk and k_0 is the typical wavenumber of the small scale backreaction, taking the explicit form

$$k_0^2 \equiv 3\omega_K^2/v_A^2\,, \tag{7}$$

- v_A is the background Alfvén velocity

- $k_0 H = \sqrt{3}\beta \equiv \frac{1}{\varepsilon}\ \beta$ must be large and so ε small in order to have small perturbations

Dimensionless quantities,

$$Y \equiv \frac{k_0 \psi_1}{\partial_{r_0} \psi_0}\,,\ \ x \equiv k_0 \left(r - r_0\right)\,,\ \ u = z/\delta\,, \tag{8}$$

where $\delta^2 = H/k_0$

Master equation

$$\partial_x^2 Y + \varepsilon \partial_u^2 Y = -\left(1 - \varepsilon u^2\right) Y\,, \tag{9}$$

$$B_z = B_{0z}\left(1 + \partial_x Y\right)\,, \tag{10}$$

$$B_r \equiv B_{1r} = -B_{0z}\sqrt{\varepsilon}\partial_u Y\,, \tag{11}$$

where the existence of linear perturbation regime requires $|Y| \ll 1$.

2 Solution of the Master equation 1.

$$Y(x,u) = \sum_{n=1}^{n=\infty} F_n(u) \sin(nx + \psi_n(u))\,, \tag{12}$$

$$2\frac{dF_n}{du}\frac{d\psi_n}{du} + F_n \frac{d^2\psi_n}{du^2} = 0\,, \tag{13}$$

$$\varepsilon \frac{d^2 F_n}{du^2} - \varepsilon F_n \left(\frac{d\psi_n}{du}\right)^2 = \left[(n^2 - 1) + \varepsilon u^2\right] F_n\,. \tag{14}$$

$$d\psi_n/du = \Theta/F_n^2\,, \tag{15}$$

$$\Theta = const\,.$$

,

$$\varepsilon\frac{d^2F_n}{du^2} - \varepsilon\frac{\Theta^2}{F_n^3} = \left[(n^2-1)+\varepsilon u^2\right]F_n\,. \tag{16}$$

$$\frac{d^2F}{du^2} - \frac{\Theta^2}{F^3} = -\left(1-u^2\right)F\,. \tag{17}$$

Boundary conditions

$$F(0) = c, F'(0) = 0$$

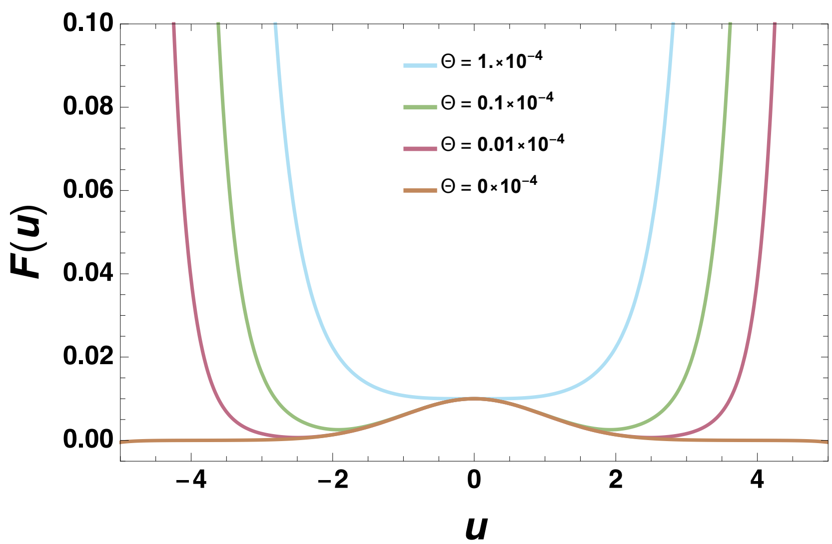

Figure 1: (Color online) Solution of 17 for $c = 0.01$ varying the parameter Θ as indicated in the plot.

3 Solution of Master equation 2

$$Y(x,u) = Y_1(x)Y_2(u)$$

$$(\frac{\partial^2}{\partial x^2}Y_1)/Y_1 = -\gamma\,, \tag{18}$$

$$\varepsilon(\frac{\partial^2}{\partial u^2}Y_2)/Y_2 + (1-\varepsilon u^2) = \gamma\,, \tag{19}$$

$$Y_1(x) = C_1 \sin(\sqrt{|\gamma|}x) + D_1 \cos(\sqrt{|\gamma|}x)\ ,\ \gamma > 0\,, \tag{20}$$

$$Y_1(x) = C_1 \sinh(\sqrt{|\gamma|}x) + D_1 \cosh(\sqrt{|\gamma|}x)\ ,\ \gamma < 0\,. \tag{21}$$

19 can be rewritten as the Kummer equation of parabolic cylinder functions using the scaling $u \to u/\sqrt{2}$. In particular, we get

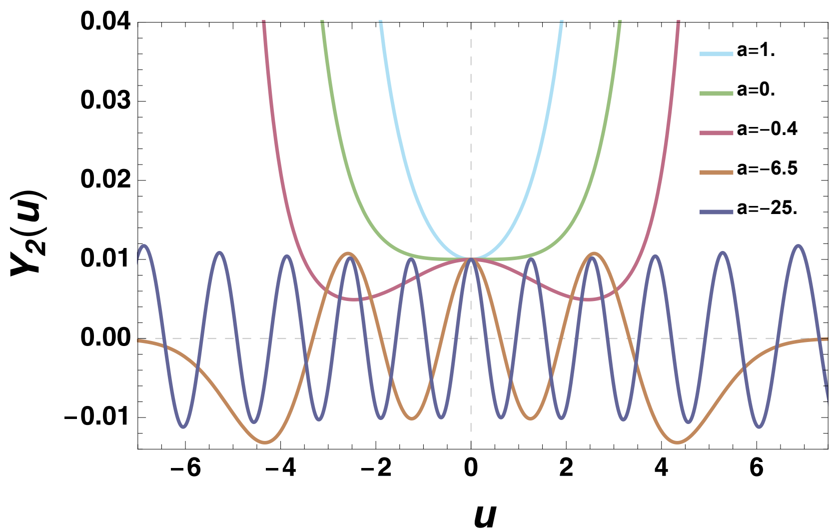

Figure 2: (Color online) Solution of 22 for $Y_2(0) = 0.01$ and $Y_2'(0) = 0$, by varying the parameter a in $[1,\ 0,\ -0.4,\ -6.5,\ -25]$ as indicated in the plot.

$$\frac{\partial^2}{\partial u^2}Y_2 - \left(\frac{1}{4}u^2 + a\right)Y_2 = 0\,, \tag{22}$$

where $a = (\gamma-1)/(2\varepsilon)$.

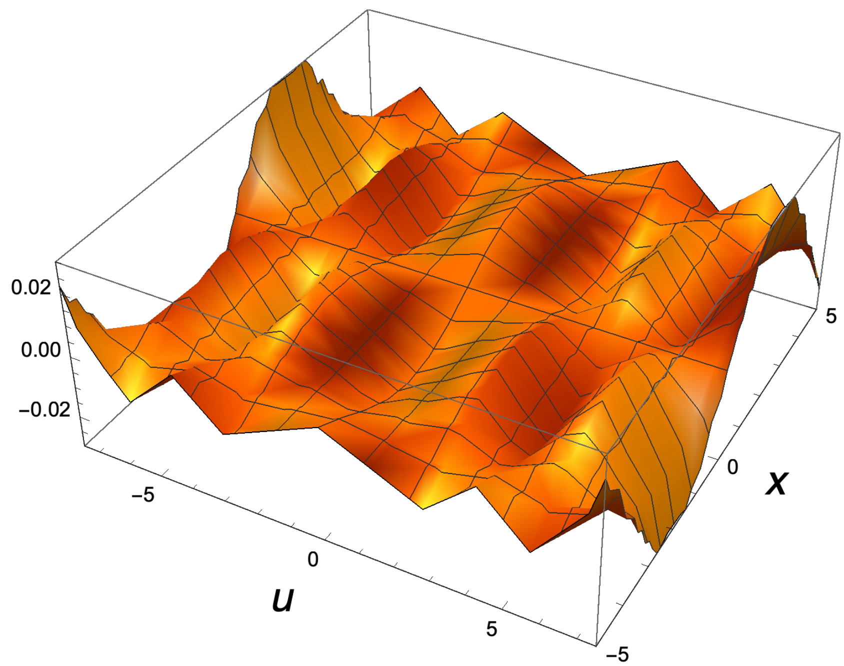

Figure 3: Plot of the solution $Y(x,u) = Y_1(x)Y_2(u)$ provided by the system of 20 (with $C_1 = 1$ and $D_1 = 0$) and 22 for $Y_2(0) = 0.01$, $Y_2'(0) = 0$, $a = -11$ (id est $\gamma = 0.78$).

4 Connection with Kummer functions and O-points

$y = u^2/2$ and $Y_2 = (2y)^{-1/4}W(y)$. 22 rewrites now as

$$\frac{d^2W}{dy^2} + \Big(-\frac{1}{4} - \frac{a}{2y} + \frac{3}{16y^2}\Big)W = 0\,, \tag{23}$$

which is a particular case of Whittaker's equation, id est

$$\frac{d^2W}{dy^2} + \Big(-\frac{1}{4} + \frac{K}{y} + \frac{1/4 - m^2}{y^2}\Big)W = 0\,, \tag{24}$$

with $K = -a/2$ and $m = \pm 1/4$. The linearly independent solutions of this equation are given by

$$W^{(1)}_{K,m} = e^{-y/2}y^{m+1/2}M(1/2 + m - K, 1 + 2m, y)\,, \tag{25}$$

$$W^{(2)}_{K,m} = e^{-y/2}y^{m+1/2}U(1/2 + m - K, 1 + 2m, y)\,, \tag{26}$$

$$Y_2^+ = u\,e^{-u^2/4}\big(c_1^+ M^+(u^2) + c_2^+ U^+(u^2)\big)\ , \tag{27}$$

$$Y_2^- = e^{-u^2/4}\big(c_1^- M^-(u^2) + c_2^- U^-(u^2)\big)\ , \tag{28}$$

where $c_{1,2}^{\pm}$ are integration constants and we denoted with +/– the choice $m = \pm 1/4$, thus

$$M^+/U^+ = M/U(a/2+3/4, 3/2; u^2/2)\ , \tag{29}$$

$$M^-/U^- = M/U(a/2+1/4, 1/2; u^2/2)\ . \tag{30}$$

$c_2^- = 0$ and $m = -1/4$ correspond to the numerical solution

$$Y_2(u) = c_1 e^{-u^2/4}\, M(a/2+1/4, 1/2; u^2/2)\ , \tag{31}$$

where c_1 is an arbitrary constant.

the zeros can be computed from the following combination of Kummer functions

$$\frac{\partial}{\partial u} Y_2(u) = -\,c_1\, uY_2/2 + 2\,c_1\, ue^{-u^2/4}(a/2+1/4)M(a/2+5/4, 3/2; u^2/2)\ . \tag{32}$$

$a = -25$, zeros of B_r t u =[0, 0.6, 1.3, 1.9, 2.55, 3.2, 3.9, ...].

$\bar{W}(a,u)$ another solution following asymptotic form for $u > 0$ and $a, \gamma \to -\infty$:

$$\bar{W}(a,u) \sim \sqrt{\pi}(4a)^{-1/2} e^{-\pi a}\big(\frac{t}{\xi^2-1}\big)^{1/2} Bi(-t)\ , \tag{33a}$$

$$\bar{W}(a,-u) \sim 2\sqrt{\pi}(4a)^{-1/2} e^{\pi a/2}\big(\frac{t}{\xi^2-1}\big)^{1/2} Ai(-t), \tag{33b}$$

$\xi = u/(2\sqrt{a})$ and $t = (4a)^{2/3}\tau$.

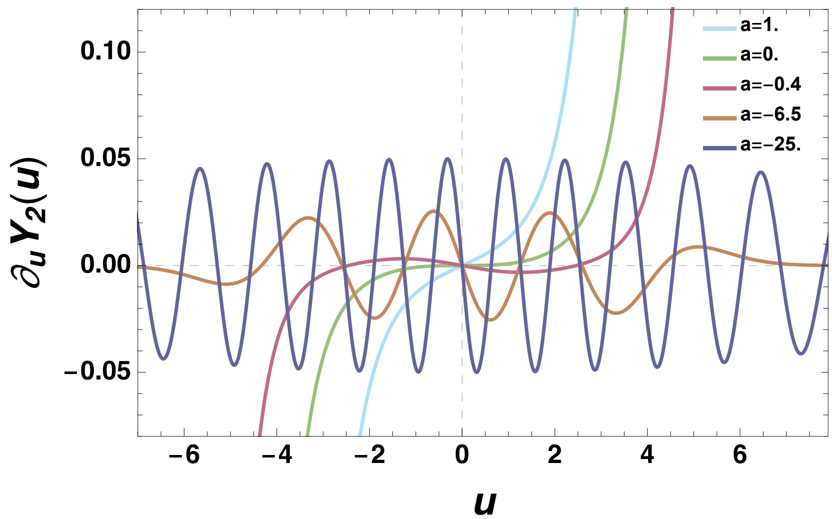

Figure 4: (Color online) Plot of the derivative of Y_2 (32), by varying the parameter a, as indicated in the plot (cf. 2). The integration constant c_1 is fixed to match the numerical solution.

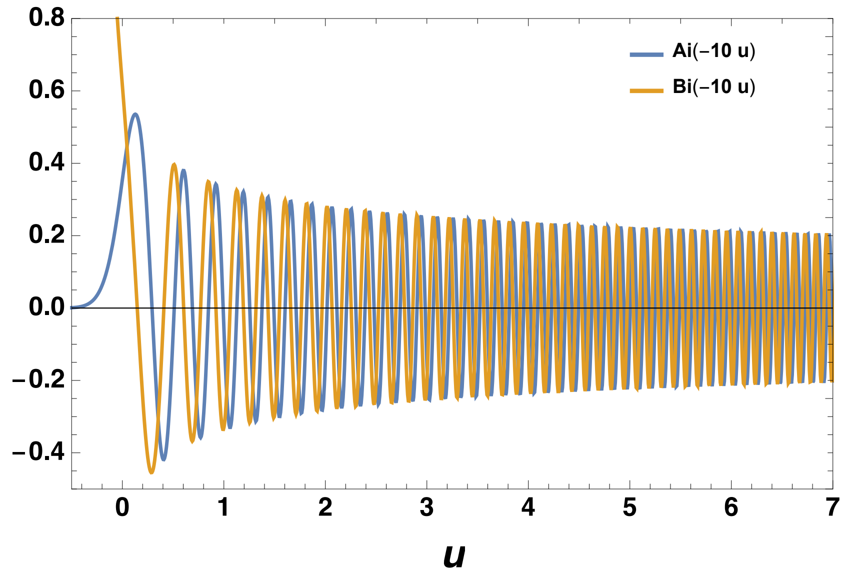

Figure 5: (color online) Plot of the function $Ai(-u)$ and $Bi(-u)$, as indicated in the plot.

$$v_z B_r - v_r B_z = J_\phi / \sigma \ , \tag{34}$$

, the accretion is suppressed, id est since $B_z \neq 0$, we must get $v_r \simeq 0$.

the plasma is quasi-ideal, we find the relation

$$v_z = v_r B_z / B_r \ . \tag{35}$$

5 Non linear theory

5.1 Governing equation

Definitions and hypothesis

- $Y = \frac{k_0 \psi}{\frac{\partial}{\partial r_0} \psi_0}$
- $D = \frac{\beta \varepsilon}{\varepsilon_0}$ definition
- $P = \beta \frac{p}{p_0}$
- $p_0 = 2 K_B T \varepsilon_0$
- $L_s = G M_S / 2 v_s^2$ definition
- $p = v_s^2 \varepsilon$, v_s sound velocity
- $\varepsilon_0 = \frac{m'}{\sqrt{r}}$ definition
- $q(r, z^2) = p(r, z^2) r$
- $A = \frac{3 G M_S m'}{\mu_0} D$

m' constant. Hypothesis:

Gravitational term is negligible with respect to the pressure gradient

$$A \psi = \frac{\partial}{\partial r} q + \frac{1}{4\pi} \frac{\partial}{\partial r} \psi \Delta \psi \, , \tag{36}$$

$$\frac{\partial}{\partial z} q + \frac{1}{4\pi} \frac{\partial}{\partial z} \psi \Delta \psi = 0 \tag{37}$$

[22]

6 Solution of the system

We separate the variables only for the magnetic flow

$$\psi(r,z) = \psi_1(r) + \psi_2(z)$$

$$A\psi = \partial_r q + \frac{1}{4\pi}\partial_r \psi(\Delta\psi) =$$

$$= \partial_r q + \frac{1}{4\pi}\psi_1(r)^{'}(\psi_1(r)^{''} + \psi_2(z)^{''})$$

$$0 = \partial_z q + \frac{1}{4\pi}\psi_2(z)'(\psi_1(r)^{''} + \psi_2(z)^{''})$$

derivate the first equation with respect to z and the second equation with respect to r, then subtracting from the first equation the second one we get a single separable equation

$$A\psi_2(z)^{'} - \frac{1}{4\pi}\psi_1^{'}(r)\psi_2^{'''}(z) + \frac{1}{4\pi}\psi_2^{'}(z)\psi_1^{'''}(r) = 0\,, \tag{38}$$

Supposing $\psi_2^{'}(z) \neq 0$ for all the values of z we obtain

$$\frac{A + \frac{1}{4\pi}\psi_1^{'''}(r)}{\frac{1}{4\pi}\psi_1^{'}(r)} = \frac{\psi_2^{'''}(z)}{\psi_2^{'}(z)}$$

$$\frac{4\pi A + \psi_1^{'''}(r)}{\psi_1^{'}(r)} = \frac{\psi_2^{'''}(z)}{\psi_2^{'}(z)} = C \tag{39}$$

where C can be any fixed constant. The two equations

$$\psi_1(r)^{'''} - C\psi_1(r)^{'} = -4\pi A$$

and

$$\psi_2(z)^{'''} - C\psi_2(z)^{'} = 0$$

are easily solved yelding

$$\psi_1(r) = \frac{D_1}{\sqrt{C}}e^{\sqrt{C}r} + \frac{4\pi A}{C}r$$

$$\psi_2(z) = \frac{D_2}{\sqrt{C}}e^{\sqrt{C}z}$$

the derivative of the pressure q satisfy the consistency condition

$$\partial_r \partial_z q = \partial_z \partial_r q$$

In fact

$$\partial_z \partial_r q = A\psi_2(z)^{'} - \frac{1}{4\pi}\psi_1(r)^{'}\psi_2^{'''}(z)$$

$$\partial_z \partial_r q = -\frac{1}{4\pi}\psi_2(z)^{'}\psi_1^{'''}(r)$$

If one substitute the expression of the third order derivatives

$$\psi_1(r)^{'''} = C\psi_1(r)^{'} - 4\pi A$$

$$\psi_2(z)^{'''} = C\psi_2(z)^{'}$$

in the expression of the mixed derivatives of the pression we immediately get that they have the same expression

$$\partial_z \partial_r q = \partial_r \partial_z q = A\psi_2(z)^{'} - \frac{C}{4\pi}\psi_2(z)^{'}\psi_1(r)^{'}$$

Let us finf now the expression for the pressure q It is also evident that no solution could be found if would have supposed that the pressure was separable as the variable ψ. Let us integrate the first equation with respect to r

$$\partial_r q = A\psi - \frac{1}{4\pi}\psi_1(r)^{'}(\psi_1(r)^{''} + \psi_2(z)^{''})$$

$$q = A(\frac{D_1}{C}e^{\sqrt{C}r} + \frac{4\pi A}{C}r^2/2) + Ar\frac{D_2}{\sqrt{C}}e^{\sqrt{C}z} -$$

$$-\frac{1}{8\pi}(D_1 e^{\sqrt{C}r} + \frac{4\pi A}{C})^2$$

$$-\frac{1}{4\pi}(D_1 D_2 e^{\sqrt{C}(r+z)} + \frac{4\pi A}{\sqrt{C}}D_2 r e^{\sqrt{C}z}) + \eta(z)$$

Now we derivate this expression for q with respect to z and insert the expression in the second equation

$$\partial_z q = -\frac{1}{4\pi}\sqrt{C}(D_1 D_2 e^{\sqrt{C}(r+z)} + \eta^{'}(z) =$$

$$= -\frac{1}{4\pi}\sqrt{C}D_1D_2e^{\sqrt{C}(r+z)} - \frac{1}{4\pi}\sqrt{C}D_2e^{2\sqrt{C}z}$$

So we get the following simple equation for $\eta(z)$

$$\eta^{'}(z) = -\frac{1}{4\pi}D_2e^{2\sqrt{C}z}\sqrt{C}$$

which gives immidiately

$$\eta(z) = -\frac{1}{8\pi}D_2e^{2\sqrt{C}z}$$

We get a close form for the pressure.

6.1 O points of ultra-nonlinear case

Since the solution $\psi_2(z)$ is an oscillating function for $C < 0$ there are an infinite number of O points, for $C > 0$ there are no such points. The The O-points in r can be found by solving the equation

$$D_1e^{\sqrt{C}r} + \frac{4\pi A}{C} = -1$$

there might be an infinite se quence or none according the values of A and C.

7 Full non linear case

We remove now the constraint that gravitational forces are neglectable with respect to pressure and solve again the pde system with variable separation we use the same symbols.

Now the equation have the form

$$\frac{\partial}{\partial z}q + \varepsilon D - 2\Delta Y\frac{\partial}{\partial z}Y = 0 \tag{40}$$

$$DY + \Delta Y(1 + \frac{\partial}{\partial r}Y) + \frac{\partial}{\partial r}P/2 = 0,, \tag{41}$$

where D, ε are given constants

We use the same variable separation

$$Y = \Psi_1(r) + \Psi_2(z)$$

The equations become

$$A : \frac{\partial}{\partial z} q + \varepsilon D - 2(\Psi_1^{''}(r) + \Psi_1^{''}(r))\Psi_2^{'}(z) = 0 \tag{42}$$

$$B : \frac{\partial}{\partial r} q + 2D(\Psi_1(r) + \Psi_2(z)) + 2(\Psi_1^{''}(r) + \Psi_1^{''}(r))(1 + \Psi_1^{'}(r)) = 0 \tag{43}$$

$$\frac{\partial}{\partial r} A - \frac{\partial}{\partial z} B = -2\Psi_1^{'''}(r)\Psi_2^{'}(z) - 2D\Psi_2^{'}(z) - 2\Psi_2^{'''}(z))(1 + \Psi_1^{'}(r)) = 0 \tag{44}$$

divide the equation by $\Psi_1^{'}(r)\Psi_2^{'}(z)$

$$\frac{\Psi_1^{'''}(r)}{\Psi_1^{'}(r)} + \frac{D}{\Psi_1^{'}(r)} + \frac{\Psi_2^{'''}(z)}{\Psi_2^{'}(z)}(1 + \Psi_1^{'}(r)) = 0 \tag{45}$$

$$\frac{1}{(1 + \Psi_1^{'}(r))}(\frac{\Psi_1^{'''}(r)}{\Psi_1^{'}(r)} + \frac{D}{\Psi_1^{'}(r)}) = -\frac{\Psi_2^{'''}(z)}{\Psi_2^{'}(z)} \tag{46}$$

getting the two equations

$$-\frac{\Psi_2^{'''}(z)}{\Psi_2^{'}(z)} = C$$

$$\frac{1}{(1 + \Psi_1^{'}(r))}(\frac{\Psi_1^{'''}(r)}{\Psi_1^{'}(r)} + \frac{D}{\Psi_1^{'}(r)}) = C$$

The first equation is easy set $\Psi_2^{'}(z) = \varphi_2(z)$ then

$$\varphi_2(z) = e^{\pm i\sqrt{C}z}$$

$$\Psi_2(z) = \mp i\frac{1}{\sqrt{C}}e^{\pm i\sqrt{C}z}$$

the other equation is a bit more triccky set $\Psi_1^{'}(r) = \varphi_1(r)$ then the equation becomes

$$\varphi_1^{''}(r) + D - C\varphi_1(1+\varphi_1) = 0 \tag{47}$$

This equation admits a constant solution $\varphi_1 = K$ such that K is a root of the equation

$$D - CK - CK^2 = 0$$

$$K = -\frac{1}{2} \pm \sqrt{1/4 + \frac{D}{C^2}}$$

We first eliminate the constant term in order to have a homogeneus diff. equation setting

$$\varphi_1 = \chi_1 + K$$

the equation is transformed in

$$\chi_1^{''} - C\chi_1(1+2K) - c\chi_1^2 = 0 \tag{48}$$

$$\chi_1^{'} = W(\chi_1) \tag{49}$$

$$\chi_1^{''}(r) = W^{'}\chi_1^{'} = W^{'}W(\chi_1)$$

inserting in the equation

$$\frac{dW}{d\chi_1}W - C(1+2K)\chi_1 - C\chi_1^2 = 0 \tag{50}$$

integrating with respect to χ_1

$$\frac{1}{2}W^2 = \frac{1}{2}C(1+2K)\chi_1^2 + \frac{C}{3}\chi_1^3 \tag{51}$$

which is equivalent to

$$\int \frac{d\chi_1}{\sqrt{(1+2K)\chi_1^2 + \frac{2}{3}\chi_1^3}} = \sqrt{C}r \tag{52}$$

the integral is equal to

$$-\frac{2}{\sqrt{1+2K}}\text{arctanh}(\frac{\sqrt{3+6K+2\chi_1}}{\sqrt{3+6K}})$$

It is easy to invert this relation and we get

$$\chi_1 = -\frac{(3+6K)}{2\cosh^2(r\frac{C(1+2K)}{2})} \tag{53}$$

Reminding that

$$\Psi^{'} = \chi_1 + K$$

we get a simple expression

$$\Psi_1(r) = -\frac{3+6K}{C(1+2K)}\tanh(\frac{1}{2}C(1+2K)r) + Kr \tag{54}$$

the complete solution is given by this simple expression

$$Y(r,z) = \Psi_1(r) + \Psi_2(z) = -\frac{3+6K}{C(1+2K)}\tanh(\frac{1}{2}C(1+2K)r) + Kr + \mp i\frac{1}{\sqrt{C}}e^{\pm i\sqrt{C}z} \tag{55}$$

According to the sign of C and K we have zeros of the radial magnetic field which are distributed as the zeros of the cosine function as it can be seen from the connecting equation

$$B_z = B_{0z}\left(1+\partial_x Y\right), \tag{56}$$

$$B_r \equiv B_{1r} = -B_{0z}\sqrt{\varepsilon}\partial_u Y, \tag{57}$$

while the xeros of the radial component satisfy the equation

$$(1+\frac{\partial}{\partial r}Y) = 0$$

$$-\frac{(3+6K)}{2\cosh^2(r\frac{C(1+2K)}{2})} + K = 0 \tag{58}$$

while the O-points in r satisfy the equation

$$r = \mathrm{arccosh}(\sqrt{\frac{3+6K}{2K}})\frac{2}{C(1+2K)} \tag{59}$$

so these points can be an infinite number or simply do not exist according to the combination of the values of C and K.

8 Full non linear case

We remove now the constraint that gravitational forces are neglectable with respect to pressure.

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