Fundamentals of Approximate Reasoning

Enric Trillas
Emeritus Researcher
European Centre for Soft Computing
Mieres (Asturias), Spain

This draft is only for the use of students in the ECSC’s Master.
## Contents

1  On the roots of fuzzy sets 11
   1.1  A genesis of fuzzy sets ............................... 11
       1.1.1 .............................................. 13
       1.1.2 .............................................. 15
       1.1.3 .............................................. 15
       1.1.4 .............................................. 16
       1.1.5 .............................................. 17
       1.1.6 .............................................. 19
   1.2  Opposite, negate, and middle ............................ 24
       1.2.1 .............................................. 24
       1.2.2 .............................................. 25
       1.2.3 .............................................. 27
       1.2.4 .............................................. 31
       1.2.5 .............................................. 33
   1.3  And/Or ............................................... 35
       1.3.1 .............................................. 35
       1.3.2 .............................................. 36
       1.3.3 .............................................. 37
   1.4  Qualified, modified, and constrained predicates .......... 37
       1.4.1 .............................................. 37
       1.4.2 .............................................. 38
       1.4.3 .............................................. 40
CONTENTS

1.4.4 .................................................. 42
1.4.5 .................................................. 43
  1.4.5.1 ............................................. 43
  1.4.5.2 ............................................. 44
1.5 Linguistic variables .................................. 45
  1.5.1 ............................................. 45
  1.5.2 ............................................. 46
  1.5.3 ............................................. 47
  1.5.4 ............................................. 50

2 Algebras of fuzzy sets .................................. 51
  2.1 Introduction ..................................... 51
    2.1.1 ........................................... 51
    2.1.2 ........................................... 52
    2.1.3 ........................................... 53
    2.1.4 ........................................... 54
    2.1.5 ........................................... 55
  2.2 The concept of an ‘algebra of fuzzy sets’ ......... 58
    2.2.1 Introduction ................................ 58
    2.2.2 Algebras of fuzzy sets ..................... 62
    2.2.3 Non-contradiction and excluded-middle ..... 66
    2.2.4 Decomposable algebras ..................... 70
    2.2.5 Standard algebras of fuzzy sets .......... 74
    2.2.6 Strong negations ......................... 80
    2.2.7 Continuous t-norms and t-conorms .......... 83
    2.2.8 Laws of fuzzy sets ....................... 86
      2.2.8.1 Distributive laws ..................... 87
      2.2.8.2 De Morgan laws ....................... 89
      2.2.8.3 Non-contradiction principle $\mu \mu' = \mu_0$ 89
      2.2.8.4 Excluded-middle principle $\mu + \mu' = \mu_1$ 90
      2.2.8.5 Both principles of Non-contradiction and Excluded-middle 90
4 Fuzzy relations
4.1 What is a fuzzy relation? .......................... 163
4.2 How to compose fuzzy relations? .......................... 165
4.3 Which relevant properties do have a fuzzy binary relation? 167
4.4 The concept of T-state .......................... 171
4.5 Fuzzy relations and \( \alpha \)-cuts .......................... 173

5 T-PREORDERS AND T-INDISTINGUISHABILITIES 179
5.1 Which is the aim of this section? .......................... 179
5.2 The characterization of T-Preorders .......................... 181
5.3 The characterization of T-indistinguishabilities .......................... 183

6 Fuzzy arithmetic 189
6.1 Introduction .......................... 189
6.1.1 ................................ 189
6.1.2 ................................ 190
6.1.3 ................................ 191
6.2 Fuzzy numbers .......................... 192
6.2.1 ................................ 192
6.2.2 ................................ 194
6.2.3 ................................ 195
6.2.4 ................................ 197
6.2.5 ................................ 198
6.2.6 ................................ 199
6.2.7 ................................ 201
6.3 A note on the lattice of fuzzy numbers .......................... 202
6.3.1 ................................ 202
6.3.2 ................................ 203
6.4 A note on fuzzy quantifiers .......................... 205
6.4.1 ................................ 205
6.4.2 ................................ 206
6.4.3 ................................ 208
CONTENTS

6.4.4 ................................................................. 209

7 Fuzzy measures ................................................. 211
  7.1 Introduction ................................................. 211
  7.2 The concept of a measure ................................. 212
  7.3 Types of measures .......................................... 215
  7.4 $\lambda$-measures ............................................ 216
  7.5 Measures of possibility and necessity ....................... 218
    7.5.1 ............................................................. 218
    7.5.2 ............................................................. 220
  7.6 Examples ..................................................... 222
  7.7 Probability, possibility and necessity ....................... 225
  7.8 Probability of fuzzy sets ................................... 227
Introduction

This book is intended as a first course’s non conventional textbook in Fuzzy Logic for engineers, and it comes from the many years in which the author taught this kind of courses as a Professor at the Technical University of Madrid, up his retirement at the end of 2006. As a consequence, the book inherits the teaching’s strategy of the author, that can be summarized by the statement ‘Nothing oughts and does substitute the own homework of the student’, from which it comes its non conventional character. Behind this strategy it is the view under which, at the university level, students and professors do learn jointly, the students do not be waiting to receive all the knowledge from the professor’s lectures, and they should read more than only the recommended textbook. For instance, neither this book is a manual with recipes to be directly applicable to practical cases, nor it is one directed to those interested in mathematical subleties. Of course, each class requires a particular tactics in the use of this textbook that, obviously, is at the hands of the professor, and depends not only on the number of lecturing hours, but on the aim of the course, and on the composition of the audience. Tutorials in which other readings, other ways of considering the topics are presented and more sophisticated problems are proposed, are essential for the learning process of the students.

This book just presents some reflections on the basic mathematical models for fuzzy logic without trying to do any subordination of it to mathematics. From the very beginning in which the student is faced with fuzzy logic it should be stressed that the main goal of this discipline is the study and com-
puter management of imprecision and its concomitant uncertainty with the best precision and certainty than possible at each case. Fuzzy logic is neither a part of mathematics, nor even of mathematical logic, like Quantum Physics is not a part of mathematics. Notwithstanding, what is without any doubt is the importance and usefulness of mathematical models in experimental sciences and technology, as they are in computer science and computer technology and, in particular, in the field of Soft Computing, where fuzzy logic plays a pivotal role. But the suitability of such models only can come from the success of its testing against some reality, for instance, in true applications. Most of what is presented in this book did prove this kind of fertility, but the students do be conscious that fuzzy logic is much more than what in this textbook.

The reader of this textbook does be acquainted with the distinguishing fact that, differently from classical logic, in fuzzy logic almost all is context dependent and purpose directed. For instance, when a system’s behavior described by means of imprecise linguistic rules is to be represented in fuzzy terms, all the necessary predicates, fuzzy connectives, and fuzzy conditionals representing the rules do be selected accordingly with its contextual meaning and with the type of inference (forwards or backwards) to be done. If in fuzzy logic everything is a matter of degree, its practice requires the art of designing the systems by means of the available theoretic armamentarium. A mistake in the design’s process can conduct to solve a different problem than the current one.

The reader does not forget that the most important in scientific and technological research is not to stop questioning (Albert Einstein), by posing good questions (Isaac Rabi), whose answers do be fertile ones (Karl Menger).
Chapter 1

On the roots of fuzzy sets

As fuzzy sets were introduced by Zadeh in 1965, they were born closely linked with imprecise predicates, that is, with names of non-precisely defined classes of objects. Even more, most of the applications of Zadeh’s ideas are made with properties the objects do verify in some degree between the two classical extremes 0 and 1, respectively. Because of that, it is not at all odd to introduce fuzzy sets from some considerations on how predicates are used in language. We will follow Wittgenstein’s statement “The meaning of a word is its use in language”.

1.1 A genesis of fuzzy sets

Usually, isolated words mean nothing. To mean something, words are to be used in a given context and in a known way. Words do serve to describe perceptions, to translate reasoning, and to expose the reasons for judgements.

For example, what it is meant by the predicate unleaty? It is impossible to answer this question, since nobody has never heard something like “this is unleaty”, “such is leaty”, etc. Neither unleaty, nor leaty, are English words, nobody has used them and they don’t appear in English dictionaries. Meaning is inherited by predicates $P$ only after being used, in some ground, by means of elemental statements ‘$x$ is $P$’. By now, unleaty has no meaning.
Words are introduced in a language by using them in concrete ways. For example, the Spanish word ‘madre’ comes from the Latin ‘mater’ that, at its turn, came from one in older Indo-European languages, but always used to name someone’s mother. Later on, the word could take, by analogy, new meanings as it is, for example, ‘mother country’, ‘goodmother’, ‘mother in law’, ‘mother of all wars’, ‘mother Nature’, ‘mother-of-pearl’, etc. The predicate *leaty* neither appears in English, nor in Spanish, French, German, ..., because it was never before used to name a property of the elements in some class, like it is with *tall*, *young*, *middle-aged*, with people, *high* with buildings or mountains, or *heavy* with metals, for example. It is only through its use that predicates do acquire ‘meaning’.

What it is meant by the predicate odd? In principle, it depends on where it is used. With natural numbers $n$, ‘$n$ is odd’ is used accordingly with the mathematical definition/rule ‘$n$ is odd if and only if once divided by 2 the rest is 1’. With people, things, or situations, odd does coincide with the meaning of ‘strange’, ‘separated’, and ‘not often’. In the first context, the predicate odd is precise or crisp, since natural numbers are only odd or not odd, but in other contexts the predicate is imprecise or fuzzy since, for example, ‘this is an odd book’, or ‘that is an odd event’, could admit degrees depending on to what extent the book or the event are odd. When the predicate is imprecise there is not a perfect classification of the objects to which it refers.

Only after a predicate does acquire meaning concepts like ‘tallness’, ‘oddity’, ‘heaviness’, etc., appear in the corresponding language. Predicates do appear in language after being used, and only after being used they can evolve in new contexts and give birth to concepts. Like it happened with ‘high’ and ‘highness’, and ‘royal highness’.

Notice that there are not natural numbers that can be qualified as ‘very odd’, or ‘more or less odd’, but there are buildings that can be called to be so. If the predicate is crisp, that is, it names just an either yes, or not, property, the application to it of a linguistic modifier needs to be newly defined, but if the predicate is imprecise it does not since we immediately understand what
it is meant.

In what follows we will only deal with predicates $P$, the name of a property, on a previously given set $X = \{x, y, z \ldots\}$, and consider the use of $P$ through the elemental statements ‘$x$ is $P$’, for all $x$ in $X$, and accepting (à la Wittgenstein) that the meaning of $P$ is its use in the current language. Hence, the first problem to afford is that placed by the following question, \textit{How the use, or meaning, of $P$ on $X$ can be mathematically represented or modeled?}

\subsection*{1.1.1}

First, and to distinguish between two statements ‘$x$ is $P$’ and ‘$y$ is $P$’, for $x \neq y$ in $X$, let us suppose it is possible to decide when is ‘$x$ is less $P$ than $y$’, that $x$ shows the property named $P$ less than $y$ shows it.

Let us call $\preceq_P$ the relation in $X$ given by

$$x \preceq_P y \iff x \text{ is less } P \text{ than } y,$$

and suppose $\preceq_P$ is a preorder (enjoys the reflexive and transitive properties). That is,

- $x \preceq_P x$, for all $x$ in $X$
- If $x \preceq_P y$, and $y \preceq_P z$, then $x \preceq_P z$.

The preordered set $(X, \preceq_P)$ reflects the organization $P$ induces in $X$, and \textit{the preorder $\preceq_P$ is the primary use of $P$ in $X$}. The relation $\preceq_P^{-1}$ is defined by ‘$x \preceq_P^{-1} y \iff y \preceq_P x$’.

We will say that ‘$x$ is equally $P$ than $y$’ whenever $x \preceq_P y$ and $y \preceq_P x$, and write $x =_P y$, with $=_P = \preceq_P \cap \preceq_P^{-1}$. Since, obviously,

- $x =_P x$, for all $x$ in $X$
- $x =_P y \iff y =_P x$
- $x =_P y$, and $y =_P z$ imply $x =_P z$,
the relation $=_{P}$ is an equivalence in $X$, and gives the quotient set $X/ =_{P}$, of the classes of equally-$P$ elements. The predicate $P$ is semi-rigid in $X$ if $X/ =_{P}$ consists in a finite number of classes. Of course, all predicates on a finite $X$ are semi-rigid.

**Example 1.1.1.** Let it be $X = \{x_1, \ldots, x_5\}$, and $P$ a predicate inducing the preorder given by the matrix with entries

$$
\text{entry}(i, j) = \begin{cases} 
1, & \text{if } x_i \leq_{P} x_j \\
0, & \text{otherwise},
\end{cases}
$$

that is,

$$
[\leq_{P}] = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1
\end{pmatrix}
$$

The quotient set $X/ =_{P}$ has the two classes $\{x_1, x_3, x_5\}$ and $\{x_2, x_4\}$. If, in the same $X$, it is $Q$ with primary use defined by

$$
x_i \leq_{Q} x_j \iff i \leq j,
$$

it results $X/ =_{Q}$ with the 5 classes $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}$.

**Example 1.1.2.** In $X = [0,10]$, consider the predicate $P$ = *around five*, with $\leq_{P}$ defined by

- If $x, y \in [0,5]$, $x \leq_{P} y \iff x \leq y$
- If $x, y \in (5,10]$, $x \leq_{P} y \iff y \leq x$
- If $x \in [0,5], y \in (5,10]$, $x$ and $y$ are not $\leq_{P}$-comparable.

Obviously, $\leq_{P}$ is a preorder, and

$$
x =_{P} y \iff \left\{ \begin{array}{l}
x \leq y \& y \leq x : x = y \\
y \leq x \& x \leq y : y = x
\end{array} \right\} \iff x = y,
$$

that is, $=_{P}$ is the equality. Hence, $X/ =_{P} = \{\{x\} : x \in [0,10]\}$, and $P$ is not semi-rigid.
1.1. A GENESIS OF FUZZY SETS

1.1.2
Let us suppose \( P \) in \( X \) given by its primary use \( \leq_p \), and let \( (L, \leq) \) be a poset, An L-degree for \( P \) in \( X \) is a function \( \mu_P : X \to L \), such that if \( x \leq_p y \), then \( \mu_P(x) \leq \mu_P(y) \)- in the order of the poset. The idea behind this definition is that \( \mu_P(x) \in L \) evaluates up to which extent \( x \) is \( P \), up to which extent \( x \) verifies the property named by \( P \). It can be written,

\[
\text{Degree up to which } x \text{ is } P = \mu_P(x) \in L,
\]

and said that the primary use of \( P \) is gradable in \( (L, \leq) \). If \( x =_p y \), then \( \mu_P(x) = \mu_P(y) \), as is easily proven. Hence, \( \mu_P \) is constant in the equivalence’s classes modulo \( =_P \).

Once \( \mu_P \) is known, it can be defined the relation \( \leq_{\mu_P} \) in \( X \) by

\[
x \leq_{\mu_P} y \iff \mu_P(x) \leq \mu_P(y),
\]

with which it is \( \leq_p \subseteq \leq_{\mu_P} \). When \( \leq_p = \leq_{\mu_P} \), \( \mu_P \) perfectly reflects the primary use of \( P \) in \( X \).

Relation \( \leq_{\mu_P} \) is a preorder since it is obviously reflexive and transitive. The pair \( (\leq_p, \leq_{\mu_P}) \) reflects a use of \( P \) in \( X \), and once \( \leq_p \) and \( (L, \leq) \) are fixed, there can exist again several uses of \( P \) in \( X \) depending on the \( L \)-degree \( \mu_P \).

1.1.3
Once a use \( (\leq_P, \leq_{\mu_P}) \) of \( P \) in \( X \) is given for the poset \( (L, \leq) \), it will be said that the new object \( P \) defined by

- \( x \in_r P \), if and only if \( r = \mu_P(x) \), for \( r \in L \),
- \( P = Q \), if and only if \( \mu_P = \mu_Q \)
is the $L$-set labeled $P$. The set $L^X = \{\mu; \mu: X \to L\}$, is usually and abusively called the set of all $L$-sets in $X$, since it contains all the possible degrees in $L$.

Notice that a predicate $P$ could give many $L$-sets, in dependance on which poset $(L, \leq)$, and which function $\mu_P \in L^X$, are chosen.

From now on, it will be supposed that $(l, \leq)$ has a minimum element $\alpha$ ($\alpha \leq r, \forall r \in L$), and a maximum element $\omega (r \leq \omega, \forall r \in L)$. With $L_0 = \{\alpha, \omega\}$, $(L_0, \leq)$ is a poset isomorphic to $([0,1], \leq)$.

Since a classical subset $A$ of $X$ is characterized by its membership function

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A, \end{cases}$$

the subset $A$ can be viewed as an $L$-set, whose function only takes the values $\alpha$ or $\omega$, that is,

$$\mu_A(x) = \begin{cases} \omega, & \text{if } x \in A \\ \alpha, & \text{if } x \notin A. \end{cases}$$

Hence, the classical subsets of $X$ are nothing else than the functions in $L_0^X$, a set included in $L^X$. Classical subsets are limiting or particular case of $L$-sets. They are degenerate $L$-sets.

Notice that if the classical subset $A$ is labeled by a crisp predicate $P$, this predicate is semi-rigid since $X/ =_P$ has, at most, two classes. Crisp predicates are rigid.

1.1.4

Very often $\mu_P$ is perceptually designed, that is, designed from what is perceived or known on the concrete use of $P$ in $X$, and it can be thought that such design is purely subjective, in the sense of being made just by what the designer believes on the use of $P$. But this would not be the case, except if the designer proceeds in a non rational way. The designer should try to be as most sure as possible on the correction of his/her/its perceptions.
In a lot of cases, mainly in the applications, predicates $P$ in $X$ result indirectly evaluable in $([0, 1], \leq)$ thanks to a numerical characteristic $Ch_P$ of the current use of $P$ in $X$. Such characteristic allows to translate ‘$X$ is $P$’ into ‘$Ch_P(x)$ is $Q$’, with an adequate predicate $Q$ on the range of the function $Ch_P : X \to R$. For example, if $X$ is a population and $P = \text{old}$, it could be the case that $Ch_{\text{old}} = \text{numerical age}=\text{Age}$, in which case ‘$x$ is old’ can be translated into ‘$\text{Age}(x)$ is big’, with a modeling of big according with the current use of old, and provided this predicate only depends on the subject’s age. In this cases, once the designer is sure that $Ch_P$ and $Q$ are good enough for the case, and also that the order in the numerical interval where $Ch_P$ ranges is adequate to model the order $\leq_P$ by the order $\leq_Q$, he/she/it should again be sure that the degrees $\mu_Q(Ch_P(x))$ agree with the expected degrees $\mu_P(x)$.

A design’s process never can arrive to something “exact”, but approximate and, if possible, keeping everything under some bounds. For example, by taking the degrees $\mu_Q(Ch_P(x))$ in intervals $(a(x), b(x))$, where $a(x)$ is the minimum of the acceptable values for $\mu_Q(Ch_P(x))$, and $b(x)$ the maximum of them. This can allow to take $\mu_Q(Ch_P(x))$ as, for instance, an average of $a(x)$ and $b(x)$. For example, if $m_x$ is the middle point of the interval $(a(x), b(x))$, and the confidence that the value is between $a(x)$ and $m_x$ can be quantified in a coefficient $a_1$, and that of being between $m_x$ and $b(x)$ by $a_2$, it can be taken $\mu_Q(Ch_P(x)) = a_1.a(x) + a_2.b(x)/a_1 + a_2$.

A rational design should be carefully made by taking into account all the available information, or knowledge, in the use of $P$ in $X$, as well as of that induced on $Q$ in its numerical universe. If the use of $P$ in $X$ is not known, it is impossible to design neither $\mu_Q(Ch_P)$ nor $\mu_P$, nor to accept $\mu_P(x) = \mu_Q(Ch_P(x))$, for all $x \in X$.

1.1.5

As is was said, if $[x]$ is a class in $X/_{=P}$, $\mu_P$ is constant in it. Let us denote by $v_x$ the value of $\mu_P$ in the class $[x]$. 
CHAPTER 1. ON THE ROOTS OF FUZZY SETS

Provided \( P \) is semi-rigid with at most two classes in \( X/ =_P \), then either \( X/ =_P = \{[x]\} \), or \( X/ =_P = \{[x], [y]\} \). In the first case, \( \mu_P \) only has a single value \( v \in L \).

In the second case, \( \mu_P \) has at most two values, and, for obvious reasons, we will only consider the situation where this values \( v_x, v_y \) are different. When,

- \( X/ =_P = \{[x]\} \), and either \( \mu_P(x) = \alpha \) for all \( x \) in \( X \), or \( \mu_P(x) = \omega \) for all \( x \) in \( X \), it results \( P = \emptyset \) in the first case, and \( P = X \) in the second. In both cases, \( P \) is a rigid or binary predicate in \( \tilde{X} \).

- \( X/ =_P = \{[x], [y]\} \), and \( v_x \neq v_y \), \( P \) is of the type

In the particular case \( \{v_x, v_y\} = \{\alpha, \omega\} \), \( P \) is of the type
that is, \( P \) is the classical or crisp subset \([x]\) of \( X\), and \( P\) is a rigid or binary predicate in \( X\).

### 1.1.6

Let us show some examples.

**Example 1.1.3.** With \( P = \text{around five} \), and \( \leq_P \) as shown in the example [2 in 1.2], the function

\[
\mu_P(x) = \begin{cases} 
0, & \text{if } x \in [0, 4] \cup [6, 10] \\
-x + 4, & \text{if } x \in [4, 5] \\
6 - x, & \text{if } x \in [5, 6], 
\end{cases}
\]

whose graphics is

verifies: \( x \leq_P y \Rightarrow \mu_P(x) \leq \mu_P(y) \), and hence \( \mu_P \) is an \([0, 1]\)—degree for \( P \) in \([0, 10]\), that originates a fuzzy set \( P \) of the numbers in \([0, 10]\) that are around five. Obviously, this degree does not perfectly reflect the primary use of \( P \).
Example 1.1.4. Consider \( P = \text{big} \) in \( X = [0, 10] \). Let us show several possible degrees for it, after agreeing that ‘\( x \leq_P y \) if and only if \( x \leq y \), in the linear order of \( [0, 10] \)’.

Hence, a \([0, 1]\)-degree \( \mu_P \) is any function \( X = [0, 10] \to [0, 1] \), such that

\[
\text{If } x \leq y, \text{ then } \mu_P(x) \leq \mu_P(y),
\]

that is, any non-decreasing function (of which there are many many). We can also agree that \( \mu_P(0) = 0 \), and \( \mu_P(10) = 1 \). With this, it is clear that all degrees for \( \text{big} \) will show some family resemblance.

Once fixed \((L, \leq) = ([0, 1], \leq)\), and \( \leq_P = \leq \), the different uses of \( \text{big} \) only depend on which function \( \mu_P \) is chosen to reflect the meaning of the predicate in \([0, 10]\). Of course, it is \( \leq_P = \leq_{\mu_P} \) if and only if function \( \mu_P \) is strictly non-decreasing, as it is the case either with \( \mu_P(x) = x \), or with \( \mu_P(x) = x^2 \). Provided \( \mu_P \) is not strictly non-decreasing as, for example, with

\[
\text{since } 6 \leq_P 4 \text{ (strictly), but } \mu_P(6) = \mu_P(4) = 0.5, \text{ } \mu_P \text{ does not perfectly reflect the primary use of } \text{big} \text{ in } [0, 10]. \text{ In the same way, the crisp degree}
\]

\[
\mu_P(x) = \begin{cases} 
1, & \text{if } x > 8 \\
0, & \text{otherwise}
\end{cases}
\]

does not perfectly reflect the primary use of \( \text{big} \) when translated into ‘after eight’. 
1.1. A GENESIS OF FUZZY SETS

Another possible model for $\mu_P$ is

$$
\mu_P(x) = \begin{cases} 
0, & \text{if } x \in [0, 2] \\
\frac{x-2}{6}, & \text{if } x \in [2, 8] \\
1 & \text{if } x \in [8, 10],
\end{cases}
$$

with graphic

All these models are linear, with the exception of $\mu_P(x) = x^2$, that is quadratic. Another quadratic models are given by $\mu_P^{(1)} = (\frac{x-2}{6})^2$, and $\mu_P^{(2)} = 1 - \mu_P^{(1)}(10 - x) = 1 - (\frac{8-x}{6})^2$, with graphics

Finally, the following two models are hyperbolic,

$$
\mu_P^{(3)}(x) = \begin{cases} 
0, & \text{if } x \in [0, 2] \\
\frac{7}{6} \frac{x-2}{x-1}, & \text{if } x \in [2, 8] \\
1 & \text{if } x \in [8, 10],
\end{cases}
$$

$$
\mu_P^{(4)}(x) = \begin{cases} 
0, & \text{if } x \in [0, 2] \\
-\frac{1}{6} (\frac{7}{x-9} + 1), & \text{if } x \in [2, 8] \\
1 & \text{if } x \in [8, 10],
\end{cases}
$$
Example 1.1.5. The predicate $old$, once numerically characterized by $Ch_{old} = Age$, can be translated into the interval $[0, 120]$, in years, by $\mu_{old}(x) = \mu_{big}(Age(x))$, and a linear model for it can be

$$
\mu_{old}(x) = \mu_{big}(Age(x)) = \begin{cases} 
0, & \text{if } 0 \leq Age(x) \leq 45 \\
\frac{x-45}{75}, & \text{if } 45 \leq Age(x) \leq 75 \\
1 & \text{if } 75 \leq Age(x) \leq 120.
\end{cases}
$$

In cases like this one, it is important to notice that the function $\mu_{old}$ depends on the values 45 and 75, as well as on the form of the curve in the sub-interval $[45, 75]$. It can be supposed, for example, that the above
1.1. A GENESIS OF FUZZY SETS

function is supplied by a person in the range of the fifties, but that one in the seventies would design \( \mu_{\text{old}} \) as the curve

![Graph showing \( \mu_{\text{old}} \) increasing quadratically between 50 and 80 years.]

with \( \mu_{\text{old}} \) increasing quadratically between the 50 and the 80 years. Analogously, a person in the twenties could design \( \mu_{\text{old}} \) as

![Graph showing \( \mu_{\text{old}} \) increasing quadratically between 45 and 75 years.]

Remark 1.1.6. There are different models for the uses of the same predicate \( P \) in \( X \), and such uses are reflected in the corresponding models \( \mu_P \) in \([0, 1]^X\). It is because of this that it is actually important the process of designing the membership functions.

Example 1.1.7. Analogously to the case of big, the predicate \( A5 = \text{around five} \) in \([0, 10]\), can have non-linear but quadratic models, as the one given by

\[
\mu_{A5}(x) = \begin{cases} 
0, & \text{if } x \in [0, 4] \cup [6, 10] \\
(x - 4)^2, & \text{if } x \in [4, 5] \\
(6 - x)^2 & \text{if } x \in [5, 6],
\end{cases}
\]

whose graphics is
Remark 1.1.8. Since each $P$ in a set $X$ can have different degrees $\mu_P$, at each particular case the meaning of $P$ should be well captured to not represent it by a mistaken function that will translate a different use of the predicate.

Remark 1.1.9. Given $\mu_P$, and the $L$–set $P$, the degree is also called the membership function of the $L$-set. At this respect,

- $x \in_\alpha P$, is classically written $x \notin \sim \bar{P}$
- $x \in_{\omega} \sim P$, is classically written $x \in \sim P$.

### 1.2 Opposite, negate, and middle

#### 1.2.1

Very often the meaning of a predicate $P$ is not captured without simultaneously capturing one of its opposites $aP$ (a for antonym, a synonym of opposite). How can I recognize that John is young without the possibility of recognizing that Peter is old? Can someone recognize that this person is tall but not than that is short?

The mastering of perception-based predicates shows this kind of polarity: we jointly learn the meaning of $P$ and some of its opposites $aP$. Even more, without knowing how to use young and old it is not possible to know how to use middle-aged, that is equivalent to ‘not young and not old’. The same could be said with respect to warm that is equivalent with ‘not cold and not
1.2. OPPOSITE, NEGATE, AND MIDDLE

'hot', in relation with water’s temperature. Composite predicates of this type are very frequent, for example *medium*, actually equivalent to ‘not big and not small’.

It should be noticed that a ‘middle’ predicate only exists with imprecise predicates, but not with precise ones. For example, in the set of natural numbers, if \( P = \text{even} \), it is \( aP = \text{odd} \), and \( a(\text{odd}) = \text{even} \), thus \( (\text{not even}) \) and \( (\text{not odd}) = \text{odd} \) and \( \text{even} \), but for no \( n \) it can be stated ‘\( n \) is odd and even’.

Let us remark that, although \( P \) and \( aP \) are linguistic terms, \( \text{not} \ P \) is not a linguistic term. For example, in all dictionary we will find \text{poor} and \text{rich}, but neither \text{not poor} nor \text{not rich}. The negate of \( P \), not \( P \), is more a logical concept than a linguistic one. Our current problem is how to find the uses of \( aP \), \( \mu_{aP} \), and \( \text{not} P(\leq_{\text{not}P}, \mu_{\text{not}P}) \), given a use \((\leq_P, \mu_P)\) of \( P \).

1.2.2

Concerning the opposite \( aP \) of \( P \), this opposition is translated by

\[ \leq_{aP} = \leq_{P}^{-1} \]

since ‘\( x \) is less \( aP \) than \( y \)’ should be equivalent to ‘\( y \) is less \( P \) than \( x \)’.

Hence, \( \leq_{a(aP)} = \leq_{aP}^{-1} = (\leq_P^{-1})^{-1} = \leq_P \), that forces \( a(aP) = P \). For example, with \( P = \text{tall} \), it is \( aP = \text{short} \) and \( a(aP) = \text{tall} \).

This property of \( aP \) shows a way for obtaining \( \mu_{aP} \) once \( \mu_P \) is known. Let it \( A : X \to X \) be a symmetry on \( X \), that is a function such that

- If \( x \leq_P y \), then \( A(y) \leq_P A(x) \)

- \( A \circ A = \text{id}_X \),

and, once \( \mu_P : X \to L \) is known, take \( \mu_{aP}(x) = \mu_P(A(x)) \), for all \( x \) in \( X \), that is, \( \mu_{aP} = \mu_P \circ A \). Function \( \mu_{aP} = \mu_P \circ A \) is a degree for \( aP \), since:

- \( x \leq_{aP} y \iff y \leq_P x \Rightarrow A(x) \leq_P A(y) \Rightarrow \mu_P(A(x)) \leq \mu_P(A(y)) \),
and verifies,

\[ \mu_{a(A)} = \mu_{(aP)} \circ A = (\mu_P \circ A) \circ A = \mu_P \circ (A \circ A) = \mu_P \circ \text{id}_X = \mu_P. \]

Then, for all symmetry \( A \) in \( X \), we have an opposite for \( P \). For example, in \( X = [0, 10] \) with \( \mu_{\text{big}}(x) = \frac{x}{10} \), and \( A(x) = 10 - x \), it is \( \mu_{\text{big}}(10 - x) = \frac{10 - x}{10} = 1 - \frac{x}{10} \), and with \( a(\text{big}) = \text{small} \), it can be said \( \mu_{\text{small}}(x) = 1 - \frac{x}{10} \).

If \( \text{big} \) is represented by

\[
\mu_{\text{big}}(x) = \begin{cases} 
0, & \text{if } x \in [0, 4] \\
\frac{x-4}{4}, & \text{if } x \in [4, 8] \\
1, & \text{if } x \in [8, 10],
\end{cases}
\]

with the same symmetry \( A(x) = 10 - x \), it results

\[
\mu_{\text{small}}(x) = \mu_{\text{big}}(10 - x) = \begin{cases} 
0, & \text{if } x \in [6, 10] \\
\frac{10-x}{4}, & \text{if } x \in [2, 6] \\
1, & \text{if } x \in [0, 2],
\end{cases}
\]

graphically,

\[ \text{It is easy to prove that } A(x) = 10 \frac{10-x}{10+x} \text{ is a symmetry in } [0, 10], \text{ since it verifies,} \]

- If \( x \leq y \), then \( A(y) \leq A(x) \),
- \( A(A(x)) = x \), for all \( x \) in \( [0, 10] \).
1.2. Opposite, Negate, and Middle

Hence, another opposite of \textit{big} with \( \mu_{\text{big}}(x) = x \), is \( \mu_{a \text{big}}(x) = \mu_{\text{big}}(A(x)) = A(x) = 10 \frac{10-x}{10+x} \). It gives a different representation for \textit{small}, \( \mu_{\text{small}}(x) = 10 \frac{10-x}{10+x} \) in [0, 1].

The last two examples show a serious trouble. There are no points \( x \in X \) such that it is simultaneously \( \mu_P(x) = 0 \) and \( \mu_{aP}(x) = 0 \). The pairs of opposites \((P, aP)\) for which there is a region in \( X \) such that both \( \mu_P \) and \( \mu_{aP} \) take the value 0, called neutral region, are called regular opposites. Hence, the two above pairs (big, small) are not regular. Nevertheless, if big is represented by

\[
\mu_{\text{big}}(x) = \begin{cases}
0, & \text{if } 0 \leq x \leq 7 \\
\frac{(x-7)}{3}, & \text{if } 7 \leq x \leq 10,
\end{cases}
\]

with \( A(x) = 10 - x \), results

\[
\mu_{\text{small}}(x) = \mu_{\text{big}}(10 - x) = \begin{cases}
0, & \text{if } 3 \leq x \leq 10 \\
\frac{(3-x)}{3}, & \text{if } 0 \leq x \leq 3,
\end{cases}
\]

with graphics

![Graph showing \( \mu_{\text{big}} \) and \( \mu_{\text{small}} \)](image)

that shows the neutral region (3, 7). This pair is regular.

1.2.3

Let it \( P \) be a predicate in \( X \), and \( P' = \text{not}P \) its negate. The only we can say about the relation between \( \leq_P \) and \( \leq_{P'} \) is that it is \( \leq_{P'} \subseteq \leq_{P}^{-1} \), since
CHAPTER 1. ON THE ROOTS OF FUZZY SETS

If $x$ is less $P$ than $y$, then $y$ is less not $P$ than $x$,
or, equivalently, $\leq_P \subset \leq_P^{-1}$. We can also easily agree that,

- If $\mu_P(x) = \alpha$, then $\mu_{P'}(x) = \omega$
- If $\mu_P(x) = \omega$, then $\mu_{P'}(x) = \alpha$.

Let it $N : L \rightarrow L$ be a function such that

1. If $a \leq b$, then $N(b) \leq N(a)$,
2. $N(\alpha) = \omega$, and $N(\omega) = \alpha$,

with such a function $N$, it is $\mu_{P'} = N \circ \mu_P$ an $L$-degree for $P'$, since

$x \leq_P y \Rightarrow y \leq_P x \Rightarrow \mu_P(y) \leq \mu_P(x) \Rightarrow N(\mu_P(x)) \leq N(\mu_P(y)) \Leftrightarrow \mu_{P'}(x) \leq \mu_{P'}(y)$.

Hence, given an $L$-degree $\mu_P$ of $P$ in $X$, with each function $N$ verifying (1) and (2), we get the $L$-degree $\mu_{P'} = N \circ \mu_P$. Such functions $N$ are called negation function.

Provided the negation function does verify

3. $N \circ N = \text{id}_L$,

then

$\mu_{(P')\gamma}(x) = N(\mu_{P'}(x)) = N(N(\mu_P(x))) = (N \circ N)(\mu_P(x)) = \text{id}_L(\mu_P(x)) = \mu_P(x),$

for all $x$ in $X$, or $\mu_{(P')\gamma} = \mu_P$.

Functions $N$ verifying (1), (2), and (3) are called strong negations, and are the only used with fuzzy sets, for the following reason. With $L = [0, 1]$, all strong negations are, obviously, continuous, hence, if $\mu_P$ is continuous also $\mu_{P'}$ is such, and if $\mu_P$ has some discontinuities, $\mu_{P'}$ has the same discontinuities. That is, strong negations do not add discontinuities to those of $\mu_P$. 
IF \( L = [0, 1] \), there is a family of strong negations widely used in fuzzy set theory, the so-called Sugeno’s negations:

\[
N_\lambda(a) = \frac{1 - a}{1 + \lambda a}, \quad \text{with } \lambda > -1, \text{ for all } a \in [0, 1].
\]

For example, \( N(a) = 1 - a, N_1(a) = \frac{1 - a}{1 + a}, N_{-0.5} = \frac{1 - a}{1 + 0.5a}, N_2(a) = \frac{1 - a}{1 + 2a} \), etc. Since obviously,

\[
N_{\lambda_1} \leq N_{\lambda_2} \iff \lambda_1 \leq \lambda_2,
\]

it results:

- If \( \lambda \in (-1, 0] \), then \( N_\lambda \leq N_0 \)

- If \( \lambda \in (0, +\infty) \), then \( N_0 < N_\lambda \),

graphically

Notice that, provided \( N \) is in the Sugeno’s family of strong negations, it is enough to know a concrete pair of numbers \( (a, N_\lambda(a)) \) to compute the corresponding \( \lambda \). For example,

- If \( N_\lambda(0.5) = 0.5 \), it results \( \lambda = 0 \)

- If \( N_\lambda(0.7) = 0.4 \), it results \( \lambda = -\frac{5}{14} \)

- If \( N_\lambda(0.4) = 0.5 \), it results \( \lambda = \frac{1}{2} \).
Example 1.2.1. With $\mu_{A5}$ as defined in 1.6.1, and with $N_0$, it is

$$
\mu_{\text{not } A5}(x) = 1 - \mu_{A5}(x) = \begin{cases} 
1, & \text{if } x \in [0, 4] \cup [6, 10] \\
5 - x, & \text{if } x \in [4, 5] \\
x - 5, & \text{if } x \in [5, 6],
\end{cases}
$$

with graphic

Example 1.2.2. With $\mu_{\text{small}}$ as defined in the second case of 2.2, with $N_1$ it is

$$
\mu_{\text{not } \text{small}}(x) = N_1(\mu_{\text{small}}(x)) = \frac{1 - \mu_{\text{small}}(x)}{1 + \mu_{\text{small}}(x)} = \begin{cases} 
1, & \text{if } 6 \leq x \leq 10 \\
\frac{x - 2}{10 - x}, & \text{if } 2 \leq x < 6 \\
0, & \text{if } 0 \leq x < 2,
\end{cases}
$$

whose graphics is
1.2. **OPPOSITE, NEGATE, AND MIDDLE**

**Example 1.2.3.** With $\mu_{\text{big}}$ as defined in the third case of 2.2, and with $N_0$, it is

$$
\mu_{\text{not \, big}}(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 7 \\
\frac{10-x}{3}, & \text{if } 7 \leq x \leq 10,
\end{cases}
$$

with graphics

![Graph](image)

1.2.4

With pairs of antonyms $(P, aP)$, it should always be taken into account that conditional statements like "If the bottle is empty, then it is not full", conduct to the inequality $\mu_{aP} \leq \mu_{\text{not } P}$, with $P = \text{full}$, showing that $\text{not } P$ is the biggest antonym of $P$. It is not often the case in which $aP = \text{not } P$, practically it only happens when $aP$ is such that there is not any linguistic term $aP$ in the language. When $aP$ and $\text{not } P$ are not coincidental, it is said that $aP$ is a strict antonym of $P$.

When modeling $\mu_{aP} = \mu_P \circ A$, and $\mu_{\text{not } P} = N \circ \mu_P$, with a symmetry $A$ of $X$ and a strong negation $N$ in $[0, 1]$, it results the condition of coherence:

$$
\mu_P \circ A \preceq N \circ \mu_P,
$$

between $A$ and $N$, that should be always verified. If $A$ is known, $N$ should be chosen to satisfy this coherence’s condition, and if what is known is $N$, then $A$ should be chosen to verify the condition.
Example 1.2.4. With $N_0(a) = 1 - a$, and $A(x) = 10 - x$ in $X = [0, 10]$, if

$$
\mu_{\text{big}}(x) = \begin{cases} 
0, & \text{if } x \in [0, 4] \\
\frac{x-4}{4}, & \text{if } x \in [4, 8] \\
1, & \text{if } x \in [8, 10],
\end{cases}
$$

results

$$
\mu_{\text{small}}(x) = \mu_{\text{big}}(10 - x) = \begin{cases} 
0, & \text{if } x \in [6, 10] \\
\frac{6-x}{4}, & \text{if } x \in [2, 6] \\
1, & \text{if } x \in [0, 2],
\end{cases}
$$

$$
\mu_{\text{not big}}(x) = 1 - \mu_{\text{big}}(x) = \begin{cases} 
1, & \text{if } x \in [0, 4] \\
\frac{8-x}{4}, & \text{if } x \in [4, 8] \\
0, & \text{if } x \in [8, 10],
\end{cases}
$$

whose graphics

show that the pair $(\text{big, small})$ is coherent, since $\mu_{\text{small}} \leq \mu_{\text{not big}}$.

Example 1.2.5. In $[0, 10]$ take $\mu_{\text{big}}(x) = \frac{x}{10}$ and $\mu_{\text{not big}}(x) = 1 - \mu_{\text{big}}(x) = 1 - \frac{x}{10}$. Which symmetries $A : [0, 10] \to [0, 10]$ can be taken for having $\mu_{\text{small}} = \mu_{\text{big}} \circ A$? From the coherence’s condition with $N_0$, follows

$$
\mu_{\text{small}}(x) = \mu_{\text{big}}(A(x)) = \frac{A(x)}{10} \leq \mu_{\text{not big}}(x) = 1 - \frac{x}{10}
$$

hence, $A(x) \leq 10 - x$ is the condition $A$ must satisfy. For example,

- If $A_1(x) = 10 - x$, it results $\mu_{\text{small}}(x) = 1 - \frac{x}{10} = \mu_{\text{not big}}(x)$, a non-regular case.
1.2. OPPOSITE, NEGATE, AND MIDDLE

- If \( A_2(x) = 10 \cdot \frac{10-x}{10+x} \), for which \( A_2(x) \leq 10 - x \), it results \( \mu_{\text{small}}(x) = \mu_{\text{big}}(10 \cdot \frac{10-x}{10+x}) = \frac{10-x}{10+x} \), with graphics.

Example 1.2.6. With the same \( \mu_{\text{big}}(x) = \frac{x}{10} \) in the previous example, and \( \mu_{\text{small}} = \mu_{\text{big}}(10-x) \), which Sugeno’s strong negation \( N_\lambda \) can be used for having \( \mu_{\text{not} \text{ big}}(x) = N_\lambda(\mu_{\text{big}}(x)) \)?

It should be,

\[
\mu_{\text{small}}(x) = \mu_{\text{big}}(10 - x) = \frac{10 - x}{10} \leq N_\lambda(\mu_{\text{big}}(x)) = N_\lambda\left(\frac{x}{10}\right) = \frac{10 - x}{10 + \lambda x}
\]

that means \( \lambda x \leq 0 \). Hence, \( \lambda \in (-1, 0] \). For example,

- With \( N_\lambda = N_0 \), there is coherence
- With \( N_\lambda = N_{-0.5} \), or \( N_\lambda(a) = \frac{2(1-a)}{2-a} \), there is coherence
- With \( N_\lambda = N_1 \), there is no coherence.

1.2.5

Given a regular pair of antonyms \( (P, aP) \), the middle or medium term of them, is the predicate \( MP = \text{not} P \ and \ Not \ aP \), as it was said in 2.1. Although we don’t yet studied the diverse models for the conjunction ‘and’, by the moment let us take the model given by the operation minimum (= min) in \( [0, 1] \). Then,

\[
\mu_{MP} = \min(\mu_{\text{not} P}(x), \mu_{\text{not} aP}(x)).
\]
For example, with $P = \text{small}$ in $[0, 10]$, $N_0$, and $A(x) = 10 - x$, is giving:

The triplets $(P, MP, aP)$ play an important role in the applications of fuzzy sets.
1.3 And/Or

Let $P$, $Q$ be two predicates in $X$. Consider the new predicates ‘$P$ and $Q$’, and ‘$P$ or $Q$’, used by means of:

- ‘$x$ is $P$ and $Q$’ $\iff$ ‘$x$ is $P$’ and ‘$x$ is $Q$’
- ‘$x$ is $P$ or $Q$’ $\iff$ Not (Not ‘$x$ is $P$’ and Not ‘$x$ is $Q$’) $\iff$ Not (‘$x$ is $P^\prime$ and ‘$x$ is $Q^\prime$) $\iff$ x is ‘($P^\prime$ and $Q^\prime$),

with respective primary uses $\leq_P \text{and} Q \subset \leq_P \cap \leq_Q$, and $\leq_P \text{or} Q$. Take L-degrees $\mu_P, \mu_Q$.

1.3.1

Given $(L, \leq)$, let $*: L \times L \to L$ be an operation, verifying the properties

- $a \leq b, c \leq d \Rightarrow a \circ c \leq b \circ d$
- $a \circ c \leq a, a \circ c \leq c$,

Then, $\mu_P(x) \circ \mu_Q(x)$ is an L-degree for $P$ and $Q$ in $X$, since:

$x \leq_{P \text{and} Q} y \Rightarrow x \leq_P y \text{ and } x \leq_Q y \Rightarrow \mu_P(x) \leq \mu_P(y) \text{ and } \mu_Q(x) \leq \mu_Q(y) \Rightarrow \mu_P(x) \circ \mu_Q(x) \leq \mu_P(y) \circ \mu_Q(y)$. Hence, defining $\mu_P(x) \circ \mu_Q(x) = \mu_{P \text{and} Q}(x)$, an L-degree for ‘$P$ and $Q$’ in $X$ is obtained, and the operation $\circ$ can be called an and-operation. Notice that it is,

$$\mu_{P \text{and} Q}(x) \leq \mu_P(x), \text{ and } \mu_{P \text{and} Q}(x) \leq \mu_Q(x).$$

In the case $(L, \leq)$ is a lattice with the minimum operation $\cdot = \text{min}$, and $*: \leq$, it implies

$$\mu_{P \text{and} Q}(x) \leq \mu_P(x) \cdot \mu_Q(x).$$

Then, if $(L, *, +, \leq)$ is a lattice, at least it is the operation $* = -(\text{min})$, with which we have the L-degree

$$\mu_{P \text{and} Q}(x) = \mu_P(x) \cdot \mu_Q(x), \forall x \in X.$$
1.3.2

If $\ast$ is an and-operation, and $N : L \to L$ is a strong negation, define

$$a \oplus b = N(N(a) \ast N(b)), \text{ for all } a, b \in L.$$ 

Since $a \leq b, c \leq d \Rightarrow N(b) \leq N(a), N(d) \leq N(c) \Rightarrow N(b) \ast N(d) \leq N(a) \ast N(c) \Rightarrow N(N(a) \ast N(c)) \leq N(N(b) \ast N(d))$, it results $a \oplus c \leq b \oplus d$. Analogously, from $N(a) \ast N(b) \leq N(a)$, it follows $a \leq N(N(a) \ast N(b)) = a \oplus b$, and $b \leq a \oplus b$, for all $a, b$.

Then, $\mu_P(x) \oplus \mu_Q(x)$ is an L-degree for $P$ or $Q$, since (remember that it is $\leq_P \subset \leq_P^{-1}$):

$$x \leq_P y \Leftrightarrow x \leq_{P', \text{ and } Q'} y \Leftrightarrow y \leq_{P' \text{ and } Q'} x \Rightarrow \mu_{P' \text{ and } Q'}(y) \leq \mu_{P' \text{ and } Q'}(x) \Leftrightarrow \mu_P(y) \ast \mu_Q(y) \leq \mu_P(x) \ast \mu_Q(x) \Rightarrow N(\mu_P(y)) \ast N(\mu_Q(y)) \leq N(\mu_P(x)) \ast N(\mu_Q(x)) \Rightarrow \mu_P(x) \oplus \mu_Q(x) \leq \mu_P(y) \oplus \mu_Q(y).$$

Hence, $\mu_P(x) \oplus \mu_Q(x)$ can be taken as an L-degree for $\text{‘}P \text{ or } Q\text{’}$ in $X$, and the operation $\oplus$ can be called an or-operation. Notice that

$$\mu_P(x) \leq \mu_{P', \text{ and } Q'}(x), \quad \mu_Q(x) \leq \mu_{P', \text{ and } Q'}(x).$$

In the case $(L, \ast, +)$ is a lattice with the maximum operation $+ = \text{max}$, and $+ \leq \ast$, it implies $\mu_P(x) + \mu_Q(x) \leq \mu_{P', \text{ and } Q'}(x)$.

Then, if $(L, \ast, +, \leq)$ is a lattice, at least it is the operation $\oplus = +(\text{max})$, with which we have the L-degree

$$\mu_{P \text{ or } Q}(x) = \mu_P(x) + \mu_Q(x), \forall x \in X.$$  

With such degree it holds the ‘duality’ law

$$\mu_{P' \text{ or } Q'} = \mu_P' \text{ and } Q.$$ 

Remark 1.3.1. The lattice operation $\cdot(\ast)$ is not necessarily the only operation $\ast(\oplus)$, that can exist. In the case $(L, \leq)$ is not a lattice, there can also exist
other operations $*$ and $\oplus$. For example, $L = [0, 1]$ is not a lattice with $* = \text{prod}$, but

$$a \leq b, c \leq d \Rightarrow \text{prod}(a, c) \leq \text{prod}(b, d),$$

and $\text{prod}(a, b) \leq a$, $\text{prod}(a, b) \leq b$. Hence, $\text{prod}$ can be eventually, used to model the use of $\text{and}$. Analogously, $\uplus = \text{prod}^*(a, b) = 1 - \text{prod}(1 - a, 1 - b) = a + b - a.b$ verifies $a \leq b, c \leq d \Rightarrow \text{prod}^*(a, c) \leq \text{prod}^*(b, d)$, and $a \leq \text{prod}^*(a, b), b \leq \text{prod}^*(a, b)$. Hence, $\text{prod}^*$ can be eventually, used to model the use of $\text{or}$.

**Remark 1.3.2.** The existence of operations $*$ and $\oplus$ in $L$, warrants the existence of $L$-degrees for and, or, respectively.

**Remark 1.3.3.** Since $a * b \leq (a * b) \oplus (a * b)$, and $a * b \leq a, a * b \leq b$ it follows

$$(a * b) \oplus (a * b) \leq a \oplus b,$$

and $a * b \leq a \oplus b$, for all $a, b$ in $L$, and all pair of operations $*$ and $\oplus$.

### 1.3.3

Given $P$, if $MP = \text{Not} P$ and Not $aP = P'$ and $(aP)'$, with $N$ for $\text{not}$, $A$ for the opposite, and $*$ for $\text{and}$, results

$$\mu_{MP}(x) = \mu_{P'} \text{ and } (aP)'(x) = \mu_{P'}(x) * \mu_{(aP)'}(x) = N(\mu_{P}(x)) * N(\mu_{P}(A(x))),$$

for all $x$ in $X$.

### 1.4 Qualified, modified, and constrained predicates

#### 1.4.1

Let $P$ be a predicate on $X$, with $L$-degree $\mu_P : X \to L$, and $\tau$ a predicate on $\mu_P(X) \subseteq L$, with $L$-degree $\mu_{\tau} : \mu_P(X) \to L$. Suppose $\leq_{\tau} \subseteq \leq_{\tau}$, and consider the qualified predicate ‘$P$ is $\tau$’,
CHAPTER 1. ON THE ROOTS OF FUZZY SETS

‘\( x \text{ is } (P \text{ is } \tau) \) := \( x \text{ is } P \text{ is } \tau \),

provided \( \emptyset \not\leq P \leq X \). On these conditions,

\[
\mu_{P \text{ is } \tau} = \mu_\tau \circ \mu_P
\]
is an \( L \)-degree for \( P \text{ is } \tau \) in \( X \), since:

\[
x \leq_{P \text{ is } \tau} y \Rightarrow x \leq_P y \Rightarrow \mu_P(x) \leq \mu_P(y) \Rightarrow \mu_P(x) \leq_\tau \mu_P(y) \Rightarrow \mu_\tau(\mu_P(x)) \leq \mu_\tau(\mu_P(y)),
\]
that is, \( (\mu_\tau \circ \mu_P)(x) \leq (\mu_\tau \circ \mu_P)(y) \).

For example, with \( L = [0,1] \), \( P = \text{small} \) in \([0,10]\), with \( \leq_P = \leq^{-1} \), and \( \mu_P(x) = 1 - \frac{x}{10} \), if \( \tau = \text{large} \) in \([0,1]\) is with \( \leq_\tau = \leq \), and

\[
\mu_\tau(x) = \begin{cases} 
0, & \text{if } 0 \leq x < 0.5 \\
1, & \text{if } 0.5 \leq x \leq 1
\end{cases}
\]
it results

\[
\mu_\tau \circ \mu_P(x) = \begin{cases} 
0, & \text{if } 5 \leq x < 0.5 \\
1, & \text{if } 0 \leq x < 5
\end{cases}
\]
that allow to the interpretation of \text{small} \text{ is} large as \text{less than five}.

Take \( \tau = \text{true} \) on \([0,1]\), with \( \leq_\tau = \leq \), and the degree \( \mu_\tau \) as a non-decreasing function \([0,1] \rightarrow [0,1] \), such that \( \mu_\tau(0) = 0 \), \( \mu_\tau(1) = 1 \), once accepting that ‘0 is \( \tau \)’ is false, and ‘1 is \( \tau \)’ is true. Taking \( \mu_\tau(x) = x \), as it is usual in fuzzy logic, it is

\[
\text{Degree up to which } x \text{ is } P \text{ is true} = \mu_\tau(\mu_P(x)) = \mu_P(x), \text{ for all } x \text{ in } X,
\]
that allows to accept

\[
\text{Degree of true of } x \text{ is } P = \mu_P(x).
\]

1.4.2

Linguistic modifiers or linguistic hedges, \( m \), are adverbs acting on \( P \) just in the concatenated form \( mP \). For example, with \( m = \text{very} \) and \( P = \text{tall} \), it is

\[
mP = \text{very tall}.
\]
1.4. QUALIFIED, MODIFIED, AND CONSTRAINED PREDICATES

A characteristic that linguistically distinguishes imprecise predicates from precise ones, is that in the first case and once $P$ and $m$ are given, $mP$ is immediately understandable. If $P$ is precise (for example, $P = \text{even}$ in the set of natural numbers), $mP$ needs of a new definition to be understandable (what it means very even?). Modifiers only modify, but do not change abruptly imprecise predicates.

If $P$ in $X$ is with the use $(\leq_P, \mu_P)$, and $m$ in $\mu_P(x) \subset L$ is with $\leq_m \leq \leq$ and $\mu_m$, provided $\leq_m \subset \leq_P$, it can be taken the degree

$$\mu_{mP} = \mu_m \circ \mu_P,$$

since, $x \leq_{mP} y \Rightarrow x \leq_P y \Rightarrow \mu_P(x) \leq \mu_P(y) \Rightarrow \mu_P(x) \leq_m \mu_P(y) \Rightarrow \mu_m(\mu_P(x)) \leq \mu_m(\mu_P(y))$, or $(\mu_m \circ \mu_P)(x) \leq (\mu_m \circ \mu_P)(y)$.

Among linguistic modifiers there are two specially interesting types:

- **Expansive modifiers**, verifying $\text{id} \mu_P(x) \leq \mu_m$,

- **Contractive modifiers**, verifying $\mu_m \leq \text{id} \mu_P(x)$.

With the expansive, it results

$$\text{id} \mu_P(x)(\mu_P(x)) = \mu_P(x) \leq \mu_m(\mu_P(x)) = \mu_mP(x): \mu_P(x) \leq \mu_mP(x), \text{ for all } x \text{ in } X.$$

With the contractive, it results

$$\mu_{mP}(x) = \mu_m(\mu_P(x)) \leq \text{id} \mu_P(x)(\mu_P(x)) = \mu_P(x): \mu_{mP}(x) \leq \mu_P(x), \text{ for all } x \text{ in } X.$$

This is what happens in $L = [0, 1]$ with the Zadeh’s old definitions,

$$\mu_{\text{more or less}}(a) = \sqrt{a}, \quad \mu_{\text{very}}(a) = a^2.$$
1.4.3

Let $P$ and $Q$ predicates in $X$ and $Y$, respectively, with uses $(\leq_P, \mu_P)$, $(\leq_Q, \mu_Q)$. Each relation $\emptyset \neq R(P, Q)$:

$$(x \text{ is } P, y \text{ is } Q) \in R(P, Q),$$

allows to define the constrained predicate $Q|P = 'Q$ if $P'$, in $X \times Y$, given by

$$(x, y) \in Q|P \iff (x \text{ is } P, y \text{ is } Q) \in R(P, Q).$$

An example is given by the interpretation

$$(x, y) \in Q|P \iff 'If x \text{ is } P, then y \text{ is } Q' \iff 'x \text{ is } P \text{ implies } y \text{ is } Q'.$$

Provided $Q|P$ induces a preorder $\leq_{Q|P}$ in $X \times Y$, and that there is an $L$-degree $\mu_{Q|P} : X \times Y \to L$,

$$(x_1, y_1) \leq_{Q|P} (x_2, y_2) \Rightarrow \mu_{Q|P}(x_1, y_1) \leq \mu_{Q|P}(x_2, y_2),$$

it could be studied how to express $\mu_{Q|P}$ by means of $\mu_P$ and $\mu_Q$.

Notice that there are several possibilities for obtaining $\leq_{Q|P}$ from both $\leq_P$ and $\leq_Q$, i.e.,

$$\leq_{Q|P} = \leq_P \times \leq_Q, \leq_{Q|P} = \leq_P^{-1} \times \leq_Q,$$

etc.

The degree $\mu_{Q|P}$ is said to be decomposable, or functionally expressible, if there is an operation $J : L \times L \to L$, such that

$$\mu_{Q|P}(x, y) = J(\mu_P(x), \mu_Q(y)),$$

for all $(x, y) \in X \times Y$, and it again remains to be tested that $\mu_{Q|P}$ is actually a $L$-degree for $Q|P$. For example,
1.4. QUALIFIED, MODIFIED, AND CONSTRAINED PREDICATES

- If $\leq_{QP} = \leq_Q \times \leq_P$, and $J$ is non-decreasing in both variables, it is
  \[(x_1, y_1) \leq_{QP} (x_2, y_2) \iff x_1 \leq_P x_2, y_1 \leq_Q y_2 \Rightarrow \mu_P(x_1) \leq \mu_P(x_2), \text{ and } \mu_Q(y_1) \leq \mu_Q(y_2) \Rightarrow J(\mu_P(x_1), \mu_Q(y_1)) \leq J(\mu_P(x_2), \mu_Q(y_2)),\]
  or
  \[\mu_Q\mu(x_1, y_1) \leq \mu_Q\mu(x_2, y_2).\]

- If $\leq_{QP} = \leq_P^{-1} \times \leq_Q^{-1}$, and $J$ is decreasing in both variables, it also follows the same conclusion,

- If $\leq_{QP} = \leq_P^{-1} \times \leq_Q$, and $J$ is decreasing in its first variable, and non-decreasing in the second, it also follows the same conclusion.

Etc.

**Remark 1.4.1.** The decomposability or functional expressibility of $\mu_Q\mu, \mu_{\not P}, \mu_aP, \mu_P$ and $Q$, and $P$ or $Q$, is not a general property of the L-degrees of the predicates $Q|P$ not $P$, aP, $P$ and $Q$, and $P$ or $Q$. What has been shown at such respect with functions $J, N, A, \ast$, and $\oplus$, respectively, is just to be taken as examples of the existence of L-degrees. Although in the applications of fuzzy logic is currently accepted that all these predicates are functionally expressible, that is, expressed through numerical functions

\[J, \ast, \oplus : [0, 1] \times [0, 1] \rightarrow [0, 1], N : [0, 1] \rightarrow [0, 1], A : X \rightarrow X,\]

it should not be considered that this is always the case.

**Remark 1.4.2.** $Q|P$ is an example of a relational predicate, that is, a predicate $R$ on $X \times Y$ such that $(x, y) \in R$, with $x \in X$, and $y \in Y$. For example, $R = \text{larger, implies, around, etc.}$ Of course, once either $x$ or $y$ are fixed, what results is a predicate (unary) in $Y$ or $X$, respectively, as it is with ‘$x$ is around $y’$, if $X = Y = [0, 10]$, where with $y = 5$ it results the unary predicate around five in $[0, 10]$. 
Relational, or binary, predicates can be either precise or imprecise. In the first case, they originate a crisp subset of $X \times Y$ defined by

$$
\mu_R(x, y) = \begin{cases} 
\omega, & \text{if } (x, y) \in R \\
\alpha, & \text{otherwise}.
\end{cases}
$$

In the second, they originate an L-set in $X \times Y$ defined by

$$
\mu_R(x, y) = \text{Degree in } L \text{ up to which it is } (x, y) \in R,
$$

once an L-degree for $R$ is known.

Remark 1.4.3. In the case $L = [0, 1]$, functions $J : [0, 1] \times [0, 1] \to [0, 1]$ allowing to represent $\mu_{Q\mid P}$ by $J \circ (\mu_P \times \mu_Q)$, are called fuzzy relations, and if the predicate $Q\mid P$ interprets a rule, these relations are called fuzzy conditionals.

1.4.4

The meaning of words is not fixed for all people and all context. For example, in a dinner with three commensals the deliciousness of the dessert plates could easily result in three different orderings of such plates. Since language is a social phenomenon, also meaning is such, and it is possible to talk on the meaning of predicates for a group of people in, of course, a given context.

For a group of people $G = \{p_1, \ldots, p_m\}$, a predicate $P$ on $X$ can show $m$ primary meanings $\preceq_{P,i}$, $1 \leq i \leq m$. Since

$$
(\bigcap_{i=1}^{m} \preceq_{P,i}) = \preceq_{P,G}
$$

is not empty (all $\preceq_{P,i}$ are reflexive), it can be taken:

*Primary meaning of $P$ on $X$ for the group $G = \preceq_{P,G}$.*

Notice that provided all $\preceq_{P,i}$ are preorders, $\preceq_{P,G}$ is also a preorder.

If $m$ L-degrees $\mu_P^{(i)}$ are known for each primary meaning $\preceq_{P,i}$, since
1.4. QUALIFIED, MODIFIED, AND CONSTRAINED PREDICATES

- \( x =_{P,G} y \iff x =_{P,1} y \& \ldots \& x =_{P,m} y, \)
- \( x \preceq_{P,G} y \iff x \preceq_{P,1} y \& \ldots \& x \preceq_{P,m} y, \)

for each function \( \Phi : L^m \to L, \) non-decreasing in each place \( i \) between 1 and \( m \) (for example, if \( a \leq b \) then \( \Phi(a, x_2, \ldots, x_m) \leq \Phi(b, x_2, \ldots, x_m) \)), or aggregation function, it results

- \( x \preceq_{P,G} y \Rightarrow \Phi(\mu^{(1)}_P(x), \ldots, \mu^{(m)}_P(x)) \leq \Phi(\mu^{(1)}_P(y), \ldots, \mu^{(m)}_P(y)), \)

that allows to take

\[
\mu^G_P(X) = \Phi(\mu^{(1)}_P(x), \ldots, \mu^{(m)}_P(x)), \quad \text{for all } x \in X,
\]

as an aggregate \( L - degree \) of \( P \) on \( X \) for the group \( G \). The meaning for \( G \) results from aggregating its people’s meanings.

1.4.5

In the language, synonymy is a complex problem whose roots are possibly to be searched for in the apparition of new facts or concepts for which there is not yet a word for their designation. Then, what is sometimes done is to designate the new fact/concept by means of an old word whose meaning is considered, for some reasons, similar to that of the new fact/concept. That is, for example, that in which the old word was already used in situations judged similar to those where the new fact/concept appears/applies.

Synonymy is related with some kind of similarity or proximity of meaning and here we will only try to present some previous treats of it.

1.4.5.1

Let \( P \) be a predicate on \( X \) with \( \leq_P \), and \( Q \) a predicate on \( Y \) with \( \leq_Q \). If there exists a bijective function \( u : X \to Y \) such that,

- \( x_1 \leq_P x_2 \iff u(x_1) \leq_Q u(x_2), \)
predicates $P$ and $Q$ are \textit{u-primary-synonyms}. Notice that when $X = Y$, with $u = id_X$, what results is that $P$ and $Q$ are $id_X$-primary synonyms, or \textit{primary synonyms} for short, if and only if $\leq_P = \leq_Q$, that is, if and only if

\begin{align*}
\text{Primary meaning of } P \text{ on } X = &\, \text{Primary meaning of } Q \text{ on } X.
\end{align*}

If $P$ and $Q$ are $id_X$-primary synonyms, it is said that they are exact or perfect synonyms when $\mu_P = \mu_Q$, and it results $(\leq_P, \leq_{\mu_P}) = (\leq_Q, \leq_{\mu_Q})$.

For example, if $P = small$ on $X = [0, 1]$ is with $\leq_P = \leq^{-1}$ (the reverse linear order on the real line), and $Q = short$ on $Y = [0, 10]$ is with $\leq_Q = \leq^{-1}$ (also the reverse linear order on the real line), $u(x) = 10x$ gives

\begin{itemize}
  \item $x \leq^{-1} y \iff 10x \leq^{-1} 10y$
\end{itemize}

taking $\leq_P \cap \leq_P^{-1}$ and $\leq_Q \cap \leq_Q^{-1}$ equal to the identity ($=$) on the real line. Then, \textit{small} and \textit{short} can be considered a pair of $u$-primary synonyms.

\subsection*{1.4.5.2}

If $P$ acts on $X$ with an $\mathcal{L}$ \textit{-degree} $\mu_P$, $Q$ acts on $Y$ and is a $u$-primary synonym of $P$, from

\begin{align*}
y_1 \leq_Q y_2 &\iff u^{-1}(y_1) \leq_P u^{-1}(y_2) \Rightarrow \mu_P(u^{-1}(y_1)) \leq \mu_P(u^{-1}(y_2)),
\end{align*}

it follows that

\begin{align*}
\mu_Q = \mu_P \circ u^{-1}
\end{align*}

is an $\mathcal{L}$ \textit{-degree} for $Q$. In this situation it is

\begin{align*}
y_1 \leq_{\mu_Q} y_2 &\iff u^{-1}(y_1) \leq_{\mu_P} u^{-1}(y_2),
\end{align*}

or

\begin{align*}
x_1 \leq_{\mu_P} x_2 &\iff u(x_1) \leq_{\mu_Q} u(x_2),
\end{align*}

that are equivalent to

\begin{align*}
\leq_{\mu_Q} = \leq_{\mu_P} \circ (u \times u).
\end{align*}
For example, with the before mentioned predicates short and small, it is

$$\mu_{\text{short}}(y) = \mu_{\text{small}}(y/10)$$

for all $y$ in $[0, 10]$, and results

$$y_1 \leq_{\mu_{\mu_{\text{Q}}}} y_2 \iff y_1/10 \leq_{\mu_{\mu_{\text{P}}}} y_2/10.$$ 

**Remark 1.4.4.** Whenever $P$ and $Q$ are $u$-synonyms, it could be stated that “$P$ means $Q$”.

**Remark 1.4.5.** The definition of primary meaning is just a formal one trying to approach an important aspect of the meaning of linguistic predicates, when acting on a given universe of discourse. The same can be said about the definition of $u$-primary synonyms with which it does not hold, in general, that a pair of linguistic synonyms are necessarily $u$—primary synonyms. Anyway, what can be said is that $Q$ is a migration of $P$ to the universe $Y$.

### 1.5 Linguistic variables

#### 1.5.1

Linguistic variables are basic tools in most application of fuzzy sets in the technology’s field. They do mainly appear when linguistically describing the behavior of the physical variables of a system. A linguistic variable explicits a concept by (linguistically) granulating some elemental components of it, by showing the perceptually distinguishable shades that are relevant for the corresponding application.

A linguistic variable $LV$ is formed after considering

1. Its principal predicate, $P$
2. One of the opposites of $P$, $aP$
3. Some linguistic modifiers $m_1, \ldots, m_n$, 

and by adding:

4. Its negate (not $P$), or the middle-predicate ($MP$), or $P$ and $Q$, or not $m_1P$, or $P$ and $m_2aP$, ...

Then $LV$, is called the linguistic variable generated by $P$, and reflects the linguistic granulation perceived for the concept. For example,

- $LV=$Age, is Age={$young$, old, middle-aged, not old, not very young,..}$
- $LV=$Truth, is Truth={$true$, false, very false, not very true,..}$
- $LV=$Temperature, is Temp={$cold$, hot, warm, not cold, not very hot,..}$
- $LV=$Size, is Size={$large$, small, medium, very large,..}$
- $LV=$Height, is
  - For buildings, Height={$low$, high, medium, very high, not very low,..}$
  - For people, Height={$tall$, short, medium, very tall, more or less short,..}$
- $LV=$Speed, is Speed={$fast$, slow, very slow, more or less fast, not fast,..}$

Although the number of terms in a Linguistic Variable can be large, usually it is comprised between 5 and 9 ($7 \pm 2$) since, in a lot of cases, less than 5 shades is poor and more than 9 is excessive. Anyway in a good number of applications there only appear the three terms $P$, $aP$, and $MP$.

1.5.2

Usually, in the applications, the variables range in the set of real numbers and, because of this, the predicate $P$ acts in some interval of the real line. For example, the linguistic variable ‘Temperature’ in the interval between
1.5. LINGUISTIC VARIABLES

-10, and 50 degrees Celsius, is often represented by only the three terms \( \mu_{\text{cold}}, \mu_{\text{hot}}, \mu_{\text{warm}} \), and with warm = not cold and not hot.

Analogously, Height for people in \([0, 2]\) meters, can be represented by the linguistic variable with the four terms \( \mu_{\text{tall}}, \mu_{\text{short}}, \mu_{\text{very short}}, \mu_{\text{more or less tall}} \) in the following figure,

1.5.3

It is sometimes useful in the applications that, once ordered in some sequence, the fuzzy sets in a linguistic variable \( LV = \{ \mu_0, \mu_1, \ldots \mu_n \} \) do form what is called a fuzzy partition (or a unit’s partition), that is, verifying

\[
\sum_{j=0}^{n} \mu_j(x) = 1, \forall x \in X.
\]
This definition is a direct generalization of what happens in the classical case. If \( X = A_0 \cup A_1 \cup \ldots \cup A_n \), with \( A_i \cap A_j = \emptyset \) for \( i \neq j \), is a classical partition of \( X \), it is \( \mu_{A_0}(x) + \mu_{A_1}(x) + \ldots + \mu_{A_n}(x) = 1 \), since for each \( x \in X \) there is just one \( A_j \) such that \( x \in A_j \), but \( x \notin A_i \), for \( i \neq j \), that is,

\[
\mu_{A_j}(x) = 1, \text{ and } \mu_{A_i}(x) = 0 \text{ if } i \neq j,
\]

that implies \( \sum_{j=0}^{n} \mu_{A_j}(x) = 1 \). Let us show three examples.

**Example 1.5.1.** In \( X = [0, 4] \), take

\[
\mu_0(x) = \begin{cases} 
1 - x, & \text{if } 0 \leq x \leq 1, \\
0 & \text{if } 1 \leq x \leq 4,
\end{cases}
\]

\[
\mu_j(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq j - 1, \\
x + 1 - j & \text{if } j - 1 \leq x \leq j, \\
j + 1 - x & \text{if } j \leq x \leq j + 1, \\
1, & \text{if } j + 1 \leq x \leq 4,
\end{cases} \quad \text{for } 1 \leq j \leq 3
\]

\[
\mu_4(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 3, \\
x - 3 & \text{if } 3 \leq x \leq 4.
\end{cases}
\]

Graphically,
• If $0 \leq x \leq 1$, \[
\sum_{j=0}^{4} \mu_j(x) = \mu_0(x) + \mu_1(x) = 1 - x + x = 1
\]

• If $1 \leq j \leq 3$, $j \leq x \leq j + 1$, \[
\sum_{j=0}^{4} \mu_j(x) = \mu_j(x) + \mu_{j+1}(x) = (j + 1 - x) + (x + 1 - j - 1) = 1.
\]

• If $3 \leq x \leq 4$, \[
\sum_{j=0}^{4} \mu_j(x) = \mu_3(x) + \mu_4(x) = 4 - x + x - 3 = 1
\]

Hence $\{\mu_0, \mu_1, \ldots, \mu_n\}$ is a fuzzy partition of $[0, 4]$. Notice that each $\mu_j$ can be labeled by the predicate around $j = A_j$, that is $\mu_j = \mu_{A_j}$.

**Example 1.5.2.** In $X = [0, 10]$, take

\[
\mu_0(x) = \begin{cases} 
1 - \frac{x}{4}, & \text{if } 0 \leq x \leq 4, \\
0, & \text{if } 4 \leq x \leq 10
\end{cases}, \quad \mu_1(x) = \begin{cases} 
\frac{x}{4}, & \text{if } 0 \leq x \leq 4, \\
1, & \text{if } 4 \leq x \leq 6, \\
\frac{10-x}{4}, & \text{if } 6 \leq x \leq 10
\end{cases}
\]

\[
\mu_2(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 6, \\
\frac{x-6}{4}, & \text{if } 6 \leq x \leq 10.
\end{cases}
\]

Graphically

Since, obviously, $\mu_0(x) + \mu_1(x) + \mu_2(x) = 1$ for all $x \in [0, 10]$, $\{\mu_0, \mu_1, \mu_2\}$ is a fuzzy partition of the interval $[0, 10]$.

**Example 1.5.3.** By proceeding in the same way that in last example, it is easy to prove that in section 1.5.2, it is

\[
\{\mu_{\text{cold}}, \mu_{\text{hot}}, \mu_{\text{war}}\},
\]

a fuzzy partition of the interval $[-10, 50]$. 
1.5.4

Notice that, with the strong negation \( N_0 = 1 - \text{id} \), if \( \{\mu_0, \mu_1, \mu_2\} \) is a fuzzy partition of \( X \), it is

\[
\begin{align*}
\mu_1(x) + \mu_2(x) &= 1 - \mu_0(x), \text{ or } \mu_1 + \mu_2 = \mu_0' \\
\mu_0(x) + \mu_1(x) &= 1 - \mu_2(x), \text{ or } \mu_0 + \mu_1 = \mu_2' \\
\mu_0(x) + \mu_2(x) &= 1 - \mu_1(x), \text{ or } \mu_0 + \mu_2 = \mu_1'
\end{align*}
\]

The complement (with \( N_0 \)) of each element in a fuzzy partition, is just the addition of the other elements in the partition.
Chapter 2

Algebras of fuzzy sets

2.1 Introduction

2.1.1

From now on it will be only considered the case in which \((L, \preceq) = ([0, 1], \preceq)\), that is, of Zadeh’s fuzzy sets, with predicates \(P\) in \(X\) only known through a degree \(\mu_P : X \to [0, 1]\), and without knowing, necessarily, its primary use \(\preceq_P\). The set of all fuzzy sets in \(X\), \([0, 1]^X\), will be also denoted by \(F(X)\). In this case, the preorder \(\leq_{\mu_P}\) is linear, or total, since for all \(x, y\) in \(X\) it is either \(\mu_P(x) \leq \mu_P(y)\), or \(\mu_P(y) \leq \mu_P(x)\), that is, it is either \(x \leq_{\mu_P} y\) or \(y \leq_{\mu_P} x\) for all \(x, y\) in \(X\). Hence, \(\leq_{\mu_P}\) rarely will perfectly reflect the primary use of \(P\) in \(X\).

In the case in which \(X\) is finite, \(X = \{x_1, \ldots, x_n\}\), the fuzzy sets \(\mu \in [0, 1]^X\), will be represented by

\[
\mu = \frac{\mu(x_1)}{x_1} + \frac{\mu(x_2)}{x_2} + \cdots + \frac{\mu(x_n)}{x_n},
\]

with the convention that if some term \(\frac{\mu(x_j)}{x_j}\) does not appear, is that it is \(\mu(x_j) = 0\). For example, with \(X = \{1, 2, 3, 4\}\), the expression

\[
\mu = 0.5/x_1 + 0.7/x_2 + 1/x_4,
\]
refers to the fuzzy set in $X$ given by $\mu(x_1) = 0.5$, $\mu(x_2) = 0.7$, $\mu(x_3) = 0$, $\mu(x_4) = 1$. Analogously, the fuzzy set $\mu' = N_0 \circ \mu$ ($N_0 = 1$—id) is

$$\mu' = 0.5/x_1 + 0.3/x_2 + 1/x_3,$$

2.1.2

If $A, B$ are crisp subsets in $X$ and $Y$, respectively, that is, $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(X)$, its cartesian product $A \times B = \{(a, b); a \in A, b \in B\} \subset X \times Y$, is with the membership function $\mu_{A \times B} : X \times Y \to \{0, 1\}$, given by

$$\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$$

for all $x \in X$, $y \in Y$. It is $\mu_{A \times B}(x, y) = 1 \Leftrightarrow \mu_A(x) = \mu_B(y) = 1$.

In the same vein, if $\mu \in F(X), \sigma \in F(Y)$, the cartesian product $\mu \times \sigma$ is defined by directly generalizing the classical case:

$$\mu \times \sigma = \min \circ (\mu, \sigma).$$

Of course, if $\mu = \mu_A, \sigma = \mu_B$, it is not only $\mu \times \sigma \in \{0, 1\}^X$ but $\mu \times \sigma = \mu_{A \times B}$.

For example, with $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}$, and

$$\mu = 1/x_1 + 0.8/x_2, \; \sigma = 0.9/y_1 + 0.7/y_2,$$

it is

$$\mu \times \sigma = 0.9/(x_1, y_1) + 0.7/(x_1, y_2) + 0.8/(x_2, y_1) + 0.7/(x_2, y_2),$$

with $(\mu \times \sigma)(x_3, y_1) = (\mu \times \sigma)(x_3, y_2) = 0$.

With $\mu_{\text{big}}(x) = \frac{x}{5}$ if $x \in [0, 5]$ and $\mu_{\text{small}}(y) = 1 - \frac{y}{7}$ if $y \in [0, 7]$, it is

$$(\mu_{\text{big}} \times \mu_{\text{small}})(x, y) = \min\left(\frac{x}{5}, 1 - \frac{y}{7}\right),$$

the representation of the cartesian product as a surface contained in the cube $[0, 5] \times [0, 7] \times [0, 1]$. 
2.1.3

If \( f : X \to Y \) is a mapping and \( A \) is a crisp subset of \( X \), \( A \subset X \), it is 
\[ f(A) = \{ y \in Y ; f(a) = y, a \in A \} \]
the f-image of \( A \) in \( Y \). Notice that 
\[ \mu_{f(A)}(y) = \sup \{ \mu_A(x) ; f(x) = y \} = \begin{cases} 1, & \text{if it exists } x \in A \text{ such that } f(x) = y, \\ 0, & \text{otherwise} \end{cases} \]
With these f-image, the mapping \( f : X \to Y \) is extended to the respective 
power sets by 
\[ \hat{f} : \mathcal{P}(X) \to \mathcal{P}(Y), \quad A \mapsto f(A). \]

In the same vein, given a mapping \( f : X \to Y \), it can be extended to the 
fuzzy power sets \( F(X), F(Y) \) by 
\[ \hat{f} : F(X) \to F(Y) \]
\[ \hat{f}(\mu)(y) = \sup \{ \mu(x) ; f(x) = y \}, \quad \text{for all } y \in Y, \]
and \( \hat{f} \) is known as the ‘extension’ of \( f \) to the fuzzy parts, and the definition 
as the Zadeh’s Extension Principle.

For example, if \( f : [0, 10] \to [0, 1] \), is given by \( f(x) = 1 - \frac{x}{10} \), the fuzzy 
set \( \mu(x) = \frac{x}{10} \) in \([0, 10]^{[0,10]} \) extends to the fuzzy set in \([0, 1] \), 
\[ \hat{f}(\mu)(y) = \sup \{ \mu(x) ; f(x) = y \} = \sup \{ \frac{x}{10} ; 1 - \frac{x}{10} = y \} = 1 - y, \quad \text{for all } y \in [0, 1]. \]
If \( X = \{1, 2, 3, 4\}, Y = \{a, b, c\}, \) the mapping \( f : X \to Y \) such that 
\[ f(1) = f(2) = a, \quad f(3) = f(4) = b, \]
extends the fuzzy set \( \mu = 1/1 + 0.4/2 + 1/3 + 0.7/4 \) in \( F(X) \), to the fuzzy set 
\( \hat{f}(\mu) \) in \( F(Y) \) with values, 
\[ \hat{f}(\mu)(a) = \max \{ \mu(x) ; x \in f^{-1}(a) \} = \max \{ \mu(1), \mu(2) \} = \max (1, 0.4) = 1 \]
\[
\hat{f}(\mu)(b) = \max\{\mu(3), \mu(4)\} = 1
\]

\[
\hat{f}(\mu)(c) = 0, \text{ since } f^{-1}(c) = \emptyset.
\]

Hence,
\[
\hat{f}(\mu) = 1/a + 1/b,
\]
that corresponds to the crisp subset \{a, b\} of \(Y\).

Notice that if \(\mu = \mu_A \in \{0, 1\}^X\), it is \(\hat{f}(\mu_A) = \mu_{f(A)}\), that is not only a crisp subset of \(Y\), but coincides with the classical extension \(f(A)\) of \(A\). Nevertheless, as it is shown by the above example, it can happen that \(\hat{f}(\mu) \in \mathcal{P}(Y)\) with \(\mu \in F(X) - \mathcal{P}(X)\).

\subsection*{2.1.4}

Like with the cartesian product and with the extension principle, all operations with fuzzy sets must reproduce, when the data are crisp, the corresponding result obtained in the crisp theory. This is the principle of preservation of the classical case, that is forced by the will, and the necessity, of including all ‘the classical’ as a particular case of the algebras of fuzzy sets.

To illustrate this preservation’s principle, let us show a negative example. With \(X = [0, 1]\), and all \(\mu \in [0, 1]^{[0,1]}\), the function
\[
\mu^*(x) = 1 - \mu(1 - x),
\]
verifies:

- \(\mu^*_0(x) = 1 - (\mu_0)(1 - x) = 1: \mu^*_0 = \mu_1\)
- \(\mu^*_1(x) = 1 - (\mu_1)(1 - x) = 1 - 1 = 0: \mu^*_1 = \mu_0\)
- \(\mu \leq \sigma \Rightarrow -\sigma(1 - x) \leq -\mu(1 - x) \Rightarrow \sigma^* \leq \mu^*\)
- \(\mu^{**}(x) = 1 - \mu^*(1 - x) = 1 - [1 - \mu(x)] = \mu(x): \mu^{**} = \mu\).
Hence, it seems that the function $\mu \mapsto \mu^*$ can be taken as a “strong negation” for the fuzzy sets in $[0, 1]$, but it is not the case. Notice that if $\mu \in \{0, 1\}^{[0,1]}$, then it should be also $\mu^* \in \{0, 1\}^{[0,1]}$, that is, if $\mu$ represents a classical subset of $[0, 1]$, also $\mu^*$ should represent not only a classical subset but precisely the complement of that represented by $\mu$. But with $A = [0, \frac{1}{2}] \subseteq X$,

$$
\mu_A(x) = \begin{cases} 
1, & 0 \leq x \leq 0.5, \\
0, & 0.5 < x \leq 1,
\end{cases}
$$

follows,

$$
\mu_A^*(x) = 1 - \mu_A(1 - x) = 1 - \begin{cases} 
1, & 0.5 < x \leq 1, \\
0, & 0 \leq x \leq 0.5,
\end{cases} = \begin{cases} 
0, & 0.5 < x \leq 1, \\
1, & 0 \leq x \leq 0.5
\end{cases}
$$

that represents the subset $[0, 0.5]$, but not $A^c = (0.5, 1]$. The unary operation $*$ violates the preservation principle, and hence it cannot be taken into account to negate fuzzy sets.

### 2.1.5

Let un denote by $\mu_r$ the constant fuzzy sets in $[0, 1]^X$, $\mu_r(x) = r$, for $r \in [0, 1]$ and all $x \in X$. Notice that in $\{0, 1\}^X$ there are only the “constants” $\mu_0$ and $\mu_1$, that correspond to the sets $\emptyset$ and $X$, respectively.

Given $\mu \in [0, 1]^X$, let us denote by $\mu^{(r)}$ the fuzzy (crisp) set

$$
\mu^{(r)}(x) = \begin{cases} 
1, & \text{if } r \leq \mu(x), \\
0, & \text{otherwise},
\end{cases}
$$

for all $r \in [0, 1]$, and by $[\mu^{(r)}]$ the corresponding classical subset $\{x \in X; r \leq \mu(x)\}$. These sets are called the $r$-cuts of $\mu$ and it is always $[\mu^{(0)}] = X$.

For example, in the following figures are shown, respectively, the constant fuzzy set $\mu_{0.5}$, and the 0.3-cut of two different fuzzy sets.
CHAPTER 2. ALGEBRAS OF FUZZY SETS

Notice that when $\mu_A \in \{0, 1\}^X$, it is

- $[\mu_A^{(0)}] = X$, since for all $x \in X$ it is $0 \leq \mu(x)$
- If $r > 0$, $[\mu_A^{(r)}] = A$, since for all $x \in A$ it is $0 < r \leq 1 = \mu_A(x)$, then, the only $r$-cuts of a crisp subset $A$ of $X$ are $X$ and $A$.

If $r \leq s$, since $s \leq \mu(x)$ implies $r \leq \mu(x)$, it results $[\mu^{(s)}] \subseteq [\mu^{(r)}]$: $r$-cuts are decreasing when the indices increase.

Let us show an example with $X = \{1, 2, 3, 4, 5\}$ and $\mu = 0.8/1 + 0.6/2 + 0.7/3 + 1/4 + 1/5$, where the only significative values for $r$ are $0.6, 0.7, 0.8$, and 1:

- $[\mu^{(0.6)}] = \{1, 2, 3, 4, 5\} = X$
2.1. INTRODUCTION

- $[\mu^{(0.7)}] = \{1, 3, 4, 5\}$
- $[\mu^{(0.8)}] = \{1, 4, 5\}$
- $[\mu^{(1)}] = \{4, 5\}$

It is clear that $0.6 \leq 0.7 \leq 0.8 \leq 1$, and $[\mu^{(1)}] \subset [\mu^{(0.8)}] \subset [\mu^{(0.7)}] \subset [\mu^{(0.6)}]$.

**Theorem 2.1.1. Theorem of resolution.** For all $\mu \in [0, 1]^X$, is $\mu(x) = \max\{r \in [0, 1]; \min(\mu_r(x), \mu^{(r)}(x))\}$, for all $x \in X$.

**Proof.** $\max_{0 \leq r \leq 1} \min(\mu_r(x), \mu^{(r)}(x)) = \max_{0 \leq r \leq 1} \min(r, \mu^{(r)}(x)) = \max_{0 \leq r \leq 1} \min\left\{ r, \begin{cases} 1, & \text{if } r \leq \mu(x), \\ 0, & \text{otherwise,} \end{cases} \right\} = \max_{0 \leq r \leq 1} \min\left\{ r, \begin{cases} 0, & \text{if } r \leq \mu(x), \\ 0, & \text{otherwise,} \end{cases} \right\} = \mu(x)$. \(\square\)

**Example 2.1.2.** With $X$ and $\mu$ in the last example, it is

$\mu(x) = \max(\min(0.6, \mu^{(0.6)}(x)), \min(0.7, \mu^{(0.7)}(x)), \min(0.8, \mu^{(0.8)}(x)), \min(1, \mu^{(1)}(x)))$,

and, for instance,

$\mu(1) = \max(\min(0.6, 1), \min(0.7, 1), \min(0.8, 1), \min(1, 0)) = 0.8$
$\mu(4) = \max(\min(0.6, 1), \min(0.7, 1), \min(0.8, 1), \min(1, 1)) = 1$

etc.

**Example 2.1.3.** With $X = \{1, 2, 3\}$, take $\mu : X \times X \rightarrow [0, 1]$, given by $\mu(i, j) = \frac{\min(i, j)}{3}$. This fuzzy set in $X \times X$ can be represented either by the matrix

$$
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{1}{3} & \frac{2}{3} & 1
\end{pmatrix},
$$

or by the graph
Since the matrices of $\mu^{(1)}$, $\mu^{(\frac{2}{3})}$, and $\mu^{(\frac{1}{3})}$, are, respectively
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\]
it results
\[
\max \left( \min \left( 1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right), \min \left( \frac{2}{3}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right), \min \left( \frac{1}{3}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) \right)
\]
\[
\max \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \frac{3}{3} & \frac{3}{3} & 1 \end{pmatrix},
\]
accordingly with the theorem of resolution.

\section*{2.2 The concept of an ‘algebra of fuzzy sets’.

\subsection*{2.2.1 Introduction}
In what follows only Zadeh’s fuzzy sets will be considered, that is, functions
\[
\mu \in F(X) = [0, 1]^X.
\]
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

These functions will be labeled only when it is some predicate $P$ in $X$ such that $\mu_P = \mu$, and it is obvious that it could be the fact of having $\mu_P = \mu_Q = \mu_R \ldots = \mu$, in which case the predicates $P, Q, R, \ldots$ are exact synonyms in $X$. Notwithstanding there are much more functions in $[0, 1]^X$ than predicates in $X$, and given a not previously labeled $\mu \in [0, 1]^X$, it can be ‘artificially’ introduced the predicate $M (= \mu)$ such that,

Degree up to which $x$ is $M = \mu(x)$, for all $x$ in $X$.

• Notice that $F(X)$ will be taken as ‘ordered’ (partially) by means of the binary pointwise relation

$$\mu \leq \sigma \iff \mu(x) \leq \sigma(x), \text{ for all } x \in X,$$

that induces the pointwise identity

$$\mu = \sigma \iff \mu \leq \sigma \text{ and } \sigma \leq \mu \iff \mu(x) = \sigma(x), \text{ for all } x \in X.$$

The pointwise relation $\leq$ is also called the ‘inclusion’, and $\mu \leq \sigma$ denoted by ‘$\mu$ is included in $\sigma$’.

It will be always considered that $F(X)$ denotes, at least, the structure $([0, 1]^X; \leq; =)$. Observe that if $\mu_A, \mu_B \in \{0, 1\}^X$, that is, $A$ and $B$ are in $P(X)$, then it follows

$$\mu_A \leq \mu_B \iff A \subset B, \mu_A = \mu_B \iff A = B,$$

and

$$x \in A \iff \mu_A(x) = 1, \ x \notin A \iff \mu_A(x) = 0.$$

The classical symbol $\in$ is the fuzzy symbol $\in_1$, and $\notin$ is $\in_0$.

For example, with the fuzzy set $P$ given by function $\mu$ in the figure it is $x \notin \sim P$ if $0 \leq x \leq 3$, and $7 \leq x \leq 10$, but $x \in \sim P$ if $4 \leq x \leq 6$, and
$x \in P$, if $x \in (3, 4) \cup (6, 7)$ with $0 < \mu(x) < 1$. If $x = 3.5$, since 
$\mu(x) = x - 3$, if $x \in (3, 4)$, it is $3.5 \in P$.

• The height of $\mu \in F(X)$ is $H(\mu) = \sup_{x \in X} \mu(x)$. In the last example, it is $H(\mu) = 1$. In the finite example

$$\mu = 0.7/x_1 + 0.9/x_2 + 0.7/x_3,$$

in $X = \{x_1, x_2, x_3, x_4\}$, it is $H(\mu) = 0.9$. In a case like

it is $H(\mu) = 1$, although there is not any $x \in R$ such that $\mu(x) = 1$. If there is some $x \in X$ such that $\mu(x) = 1$, it is said that $\mu$ is a normalized fuzzy set.

• In the case $X$ is finite, $X = \{x_1, \ldots, x_n\}$, the number 

$$|\mu| = \sum_{i=1}^{n} \mu(x_i)$$

is the crisp-cardinality, or sigma-count, of $\mu$, a name coming from the fact that if $A \subset X$ has $p$ elements it is $\sum_{i=1}^{n} \mu(x_i) = p$. Obviously,
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

$\mu_\varnothing = \mu_0$, gives $|\mu_0| = 0$, $\mu_X = \mu_1$, gives $|\mu_1| = n$, and, $\mu \leq \sigma$ implies $|\mu| \leq |\sigma|$.

Remark 2.2.1. The pointwise definition of fuzzy sets inclusion implies that, for example, the fuzzy sets

$$
\mu = 0.7/x_1 + 0.8/x_2 + 1/x_3 + 0.7/x_4
$$
$$
\sigma = 0.70001/x_1 + 0.7/x_2 + 1/x_3 + 0.6/x_4
$$

in $X$, do not verify $\mu \leq \sigma$ although it is $\sigma(x_2) < \mu(x_2)$, $\sigma(x_3) = \mu(x_3)$, $\sigma(x_4) < \mu(x_4)$, but $\sigma(x_1) > \mu(x_1)$, with $\sigma(x_1) - \mu(x_1) = 0.00001$. Pointwise ‘inclusion’ is strongly affected by very small variations of the membership values. Actually, it is not a flexible, or fuzzy, concept, but a crisp one.

Because of this, it could be preferable to take the inclusion of fuzzy sets as an gradable concept $\leq_r$ $(r \in [0,1])$, and a used definition of which is

$$
\mu \leq_r \sigma \iff \frac{|\min(\mu, \sigma)|}{|\mu|} \leq r,
$$

with $|\min(\mu, \sigma)| = \sum_{i=1}^n \min(\mu(x_i), \sigma(x_i))$.

In last example, it is $|\min(\mu, \sigma)| = 0.7 + 0.7 + 1 + 0.6 = 3$, $|\mu| = 0.7 + 0.8 + 1 + 0.7 = 3.2$, and $r = 3/3.2 = 0.9375 \approx 0.94$. That is, $\mu \leq_{0.9375} \sigma$: $\mu$ “is almost included in ” $\sigma$.

Since $|\sigma| = 0.70001 + 0.7 + 1 + 0.6 = 3.00001$, it is $\frac{|\min(\mu, \sigma)|}{|\sigma|} = 0.9999$, or $\sigma \leq_{0.9999} \mu$, that is, $\sigma$ is more included in $\mu$, than $\mu$ is included in $\sigma$.

Remark 2.2.2. Of course, if $\mu \leq \sigma$, it is $\min(\mu, \sigma) = \mu$, and $r = 1$, that is,

$$
\mu \leq \sigma \Rightarrow \mu \leq_1 \sigma.
$$

Nevertheless, since it is only

$$
\sum \min(\mu(x_i), \sigma(x_i)) \leq \min(\sum \mu(x_i), \sum \sigma(x_i)),
$$

from $\mu \leq_1 \sigma$ (or $|\min(\mu, \sigma)| \leq |\mu|$) it does not necessarily follow $\mu \leq \sigma$. 
Let us show an example with crisp subsets. If \( X = \{1, 2, 3, 4, 5, 6, 7\} \), and \( A = \{1, 3, 5, 7\} \), \( B = \{1, 3, 6\} \), it is
\[
\sum_{i=1}^{7} \min(\mu_A(i), \mu_B(i)) = 3, \quad \sum_{i=1}^{7} \mu_B(i) = 4, \quad \sum_{i=1}^{7} \mu_A(i) = 4.
\]
hence
\[
\mu_A \leq \frac{3}{4} \mu_B, \text{ or } A \subseteq \frac{3}{4} B
\]
\[
\mu_B \leq \frac{3}{4} \mu_A, \text{ or } B \subseteq \frac{3}{4} A
\]

### 2.2.2 Algebras of fuzzy sets

Once \( F(X) = ([0, 1]^X; \leq; =) \) is taken, a general algebra of fuzzy sets comes from endowing \( F(X) \) with three operations:

1. \( \cdot : [0, 1]^X \times [0, 1]^X \to [0, 1]^X \),
2. \( + : [0, 1]^X \times [0, 1]^X \to [0, 1]^X \),

respectively called the complement \( \mu' \) of \( \mu \), the intersection \( \mu \cdot \sigma \) of \( \mu \) and \( \sigma \), and the union \( \mu + \sigma \) of \( \mu \) and \( \sigma \). Then \( ([0, 1]^X; \leq; =; +') \), is called an algebra of fuzzy sets, provided the following laws do hold:

a) If \( \mu \leq \sigma \), then \( \gamma \cdot \mu \leq \gamma \cdot \sigma \), and \( \mu \cdot \gamma \leq \sigma \cdot \gamma \), for all \( \gamma \in [0, 1]^X \)

b) If \( \mu \leq \sigma \), then \( \mu + \gamma \leq \sigma + \gamma \), and \( \gamma + \mu \leq \gamma + \sigma \) for all \( \gamma \in [0, 1]^X \)

c) If \( \mu \leq \sigma \), then \( \sigma' \leq \mu' \)

d) For any \( \mu \in [0, 1]^X \), \( \mu \cdot \mu_1 = \mu_1 \cdot \mu = \mu \), \( \mu + \mu_0 = \mu_0 + \mu = \mu \)

e) For all \( \mu_A, \mu_B \in [0, 1]^X \), \( \mu_A' = \mu_A \cdot \mu_B = \mu_A \land B \), \( \mu_A + \mu_B = \mu_A \lor B \) (preservation of the classical case).
Remark 2.2.3. It is not difficult to prove that no algebra of fuzzy sets is a boolean algebra. The proof comes from the fact that to be a boolean algebra would imply $\mu \cdot \mu' = \mu_0$ and $\mu + \mu' = \mu_1$ for all $\mu \in [0,1]^X$, and consists in finding some $\mu$ for which these equalities are not satisfied.

Remark 2.2.4. It is immediate to prove that $\mu \cdot \mu_0 = \mu_0 \cdot \mu = \mu_0$, $\mu + \mu_1 = \mu_1 + \mu = \mu_1$ for all $\mu \in [0,1]^X$.

Remark 2.2.5. Notice that the laws $\mu \cdot \sigma = \sigma \cdot \mu$ (commutativity of the intersection), $\mu + \sigma = \sigma + \mu$ (commutativity of the union), and $\mu'' = \mu$ (involution of the complement) are not supposed to be always verified. Nor it is supposed that the algebras $([0,1]^X,+,1)$ are dual ones, that is, the so-called De Morgan laws,

$$(\mu + \sigma)' = \mu' \cdot \sigma', (\mu \cdot \sigma)' = \mu' + \sigma',$$

are not supposed to hold in general. It is also not supported $\mu \cdot \mu = \mu$, and $\sigma + \sigma = \sigma$, etc.

Theorem 2.2.6. For any $\mu, \sigma \in [0,1]^X$: $\mu \cdot \sigma \leq \min(\mu, \sigma) \leq \max(\mu, \sigma) \leq \mu + \sigma$.

Proof. From $\mu \leq \mu_1 (\mu(x) \leq 1$, for all $x \in X)$, follows $\mu \cdot \sigma \leq \mu_1 \cdot \sigma = \sigma$. From $\sigma \leq \mu_1$, follows $\mu \cdot \sigma \leq \mu_1 \cdot \mu = \mu$. Thus,

$$(\mu \cdot \sigma)(x) \leq \sigma(x), (\mu \cdot \sigma)(x) \leq \mu(x) \Rightarrow (\mu \cdot \sigma)(x) \leq \min(\mu(x), \sigma(x)) = \min(\mu, \sigma)(x),$$

or $\mu \cdot \sigma \leq \min(\mu, \sigma)$. Hence, the operation min is the greatest possible intersection of fuzzy sets. Analogously, $\mu_0 \leq \mu, \mu_0 \leq \sigma \Rightarrow \mu_0 + \sigma = \sigma \leq \mu + \sigma$, $\mu + \mu_0 = \mu \leq \mu + \sigma$, and $\max(\mu, \sigma) \leq \mu + \sigma$: The operation max gives the smallest possible union of fuzzy sets. □

Obviously, for all $\mu, \sigma \in [0,1]^X$:

$$\mu \cdot \sigma \leq \mu \leq \mu + \sigma, \mu \cdot \sigma \leq \sigma \leq \mu + \sigma,$$

and $\mu \cdot \sigma \leq \min(\mu, \sigma) \leq \max(\mu, \sigma) \leq \mu + \sigma$. 
Theorem 2.2.7. An operation $*: [0, 1]^X \times [0, 1]^X \rightarrow [0, 1]^X$ is called idempotent if and only if $\mu * \mu = \mu$, for all $\mu \in [0, 1]^X$

- The intersection $\cdot$ is idempotent if and only if $\cdot = \min$
- The union $+$ is idempotent if and only if $+ = \max$

Proof. The operations $\min$, and $\max$ are obviously idempotent. Let us show that if $\cdot$ is idempotent, it must be $\cdot = \min$. By theorem 2.2.6, it is always $\mu \cdot \sigma \leq \min(\mu, \sigma)$, and the idempotency of $\cdot$ implies

$$\min(\mu, \sigma) \cdot \min(\mu, \sigma) = \min(\mu, \sigma).$$

But $\min(\mu, \sigma) \leq \mu, \min(\mu, \sigma) \leq \sigma$, imply $\min(\mu, \sigma) \cdot \min(\mu, \sigma) \leq \mu \cdot \sigma$, that is

$$\min(\mu, \sigma) \leq \mu \cdot \sigma,$$

and, by theorem 2.2.6, $\min(\mu, \sigma) = \mu \cdot \sigma$. A similar proof applies to $+$ and $\max$.  

Theorem 2.2.8. (Absorption laws)

- $\mu \cdot (\mu + \sigma) = \mu$ holds for all $\mu, \sigma \in [0, 1]^X \Leftrightarrow \cdot = \min$
- $\mu + (\mu \cdot \sigma) = \mu$ holds for all $\mu, \sigma \in [0, 1]^X \Leftrightarrow + = \max$

Proof. If $\cdot = \min$, the formula $\min(\mu, \mu + \sigma) = \mu$ does hold, since $\mu \leq \mu + \sigma$. Provided it is always $\mu \cdot (\mu + \sigma) = \mu$, taking $\sigma = \mu_0$ follows $\mu \cdot (\mu + \mu_0) = \mu \cdot \mu = \mu$, that holds if and only if $\cdot = \min$. If $+ = \max$, the formula $\max(\mu, \mu \cdot \sigma) = \mu$ does hold since $\mu \cdot \sigma \leq \mu$. Provided it is always $\mu + (\mu \cdot \sigma) = \mu$, taking $\sigma = \mu_1$ follows $\mu + (\mu \cdot \mu_1) = \mu + \mu = \mu$, that holds if and only if $+ = \max$.

Theorem 2.2.9. (Duality, or De Morgan laws) Provided the complement $'$ is involutive ($(\mu')' = \mu'' = \mu$, for all $\mu \in [0, 1]^X$), the algebra of fuzzy sets $([0, 1]^X, \min, \max, ')$ is a dual algebra.
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

Proof. If ′ is involutive, from \( \mu' + \sigma' = (\mu \cdot \sigma)' \) it follows \( \mu' \cdot \sigma' = (\mu + \sigma)' \) since \( \mu + \sigma = \mu'' + \sigma'' = (\mu' \cdot \sigma')' \), hence \( (\mu + \sigma)' = \mu' \cdot \sigma' \). The converse is proven in the same way. And the two De Morgan laws

\[
(\mu \cdot \sigma)' = \mu' + \sigma', (\mu + \sigma)' = \mu' \cdot \sigma'
\]

result equivalent. Then, it is enough to prove \( \max(\mu, \sigma) = (\min(\mu', \sigma'))' \), for all \( \mu, \sigma \) in \([0, 1]^X\). Since

\[
\min(\mu', \sigma') \leq \mu', \min(\mu', \sigma') \leq \sigma',
\]

it follows

\[
\mu = \mu'' \leq (\min(\mu', \sigma'))', \sigma = \sigma'' \leq (\min(\mu', \sigma'))',
\]

and

\[
\max(\mu, \sigma) \leq (\min(\mu', \sigma'))' \tag{2.1}
\]

On the other hand,

\[
\mu \leq \max(\mu, \sigma) \Rightarrow (\max(\mu, \sigma))' \leq \mu', \quad \sigma \leq \max(\mu, \sigma) \Rightarrow (\max(\mu, \sigma))' \leq \sigma',
\]

imply \( (\max(\mu, \sigma))' \leq \min(\mu', \sigma') \), or

\[
(\min(\mu', \sigma'))' \leq \max(\mu, \sigma). \tag{2.2}
\]

Now, from (2.1) and (2.2), follows the result. □

Theorem 2.2.10. (Kleene’s Law) In all algebra of fuzzy sets it holds the law

\[
\mu \cdot \mu' \leq \sigma + \sigma',
\]

for all \( \mu, \sigma \) in \([0, 1]^X\).

Proof. We have to prove that, for any \( x \in X \), it is \( (\mu \cdot \mu')(x) \leq (\sigma + \sigma')(x) \). But it only can be either \( \mu(x) \leq \sigma(x) \), or \( \sigma(x) \leq \mu(x) \) for each \( x \in X \). In the first case, it is \( (\mu \cdot \mu')(x) \leq \min(\mu(x), \mu'(x)) \leq \mu(x) \leq \sigma(x) \leq \max(\sigma(x), \sigma'(x)) = (\max(\sigma, \sigma')(x) \leq (\sigma + \sigma')(x) \). In the second case, it is \( \mu'(x) \leq \sigma'(x) \), and \( (\mu \cdot \mu')(x) \leq \min(\mu(x), \mu'(x)) \leq \mu'(x) \leq \sigma'(x) \leq \max(\sigma(x), \sigma'(x)) = (\max(\sigma, \sigma')(x) \leq (\sigma + \sigma')(x). □
Remark 2.2.11. Concerning duality, theorem 2.2.9 only states that the algebra given by the triplets \((\text{min, max, }')\), with \(\text{'}\) involutive, are dual algebras. But they are not the only dual algebras. For example, with \(\cdot = \) product,

\[
(\mu \cdot \sigma)(x) = \mu(x) \cdot \sigma(x), \forall x \in X,
\]

it is easy to proof that taking \(\mu'(x) = 1 - \mu(x)\), and

\[
(\mu + \sigma)(x) = 1 - (1 - \mu(x))(1 - \sigma(x)) = \mu(x) + \sigma(x) - \mu(x).\sigma(x),
\]

it is \(([0, 1]^X, \cdot, +,'\)\) an algebra of fuzzy sets that with

\[
\mu + \sigma = (\mu' \cdot \sigma')',
\]

is a dual algebra. Nevertheless, with \(\mu'(x) = 1 - \mu(x)\), \((\mu \cdot \sigma)(x) = \mu(x).\sigma(x)\), and \((\mu + \sigma)(x) = \max(\mu(x), \sigma(x))\), we get an algebra that is not dual since

\[
(\mu' \cdot \sigma')'(x) = \mu(x) + \sigma(x) - \mu(x).\sigma(x)
\]

does not coincide with \(\max(\mu(x), \sigma(x))\), as it is easy to see.

Remark 2.2.12. It is easy to prove that, for each algebra of fuzzy sets \(([0, 1]^X, \cdot, +,'\)\), the operation

\[
\mu +' \sigma = (\mu' \cdot \sigma')',
\]

gives the new algebra \(([0, 1]^X, \cdot, +,'')\). If the complement \(\text{'}\) is involutive \((\mu'' = \mu)\), then \((\mu +' \sigma)' = \mu' \cdot \sigma'\).

Analogously, with the operation \(\mu' \sigma = (\mu +' \sigma')'\), one has the new algebra \(([0, 1]^X, , , +,'')\) and, if \(\text{'}\) is involutive, \((\mu' \sigma)' = \mu +' \sigma'\)

2.2.3 Non-contradiction and excluded-middle

An statement is self-contradictory whenever entails its negation. For example, the only classical set that is self-contradictory is the empty one:

\[
A \subset A^c \Rightarrow A \cap A \subset A \cap A^c \Rightarrow A \subset \emptyset \Rightarrow A = \emptyset.
\]
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

Perhaps, this is the reason of the difficulty children do have on accepting that \( \emptyset \) is a set!

Within an algebra of fuzzy sets there are many many self-contradictory fuzzy sets. For example, with \( N = 1 - \text{id} \) it is

\[
\mu \leq \mu' \text{ if and only if } \mu(x) \leq 1 - \mu(x) \Leftrightarrow \mu(x) \leq \frac{1}{2}, \forall x \in X,
\]

hence: \( \mu \) is self-contradictory if and only if \( \mu \leq \frac{1}{2} \). Analogously, with the strong negation \( N(x) = \frac{1-x}{1+x} \), it is

\[
\mu \leq \mu' \Leftrightarrow \mu(x) \leq \frac{1-\mu(x)}{1+\mu(x)} \Leftrightarrow \mu(x)^2 + 2\mu(x) - 1 \leq 0 \Leftrightarrow \mu(x) \leq \sqrt{2} - 1, \forall x \in X,
\]

that is, \( \mu \) is self-contradictory if and only if \( \mu \leq \sqrt{2} - 1 \).

Notice that \( 1/2 \) is the fixed-point of the strong negation \( N = 1 - \text{id} \)

\[
(1 - n = n \Leftrightarrow n = 1/2),
\]

and that \( \sqrt{2} - 1 \) is the fixed-point of the strong negation \( N = \frac{1-\text{id}}{1+\text{id}} \)

\[
(\frac{1-n}{1+n} = n \Leftrightarrow n = \sqrt{2} - 1).
\]

The classical principle of non-contradiction, “there is impossible to have both an statement and its negation”, could be interpreted as “P and not P is impossible”, or “P and not P is self-contradictory”. All algebras of fuzzy sets do verify the principle of non-contradiction once stated in this form.

**Theorem 2.2.13.** If \( ([0,1]^X, \cdot, +, \prime) \) is an algebra of fuzzy sets, it holds the principle of non-contradiction stated by: \( \mu \cdot \mu' \leq (\mu \cdot \mu')' \) for all \( \mu \in [0,1]^X \). That is, for all \( \mu \in [0,1]^X, \mu \cdot \mu' \) is self-contradictory.

**Proof.** It is \( \mu \cdot \mu' \leq \min(\mu, \mu') \leq \mu \), and \( \mu \cdot \mu' \leq \min(\mu, \mu') \leq \mu' \); from the first inequality it follows \( \mu' \leq (\mu \cdot \mu')' \), and then the second implies \( \mu \cdot \mu' \leq (\mu \cdot \mu')' \). \( \Box \)

Notice that no additional hypotheses on the connective \( \cdot \), and the complement \( \prime \), are needed for the proof of this theorem. In the algebras of fuzzy sets the non-contradiction principle is a theorem: the algebra’s axioms imply
the principle. It is not true, as it is sometimes stated, that fuzzy sets do not verify the principle of non-contradiction in which science is based.

The classical principle of Excluded-Middle, “It is always P or not P”, can be interpreted as “Not (P or Not P) is a self-contradiction” and it is verified by all algebra of fuzzy sets.

**Theorem 2.2.14.** If \([0,1]^{X}, \cdot, +, ' \) is an algebra of fuzzy sets, it holds the principle of Excluded-Middle stated by: \((\mu + \mu ')' \leq ((\mu + \mu ')')' \) for all \(\mu \in [0,1]^{X} \). That is, for all \(\mu \in [0,1]^{X}, (\mu + \mu ')' \) is self-contradictory.

**Proof.** It is,

- \(\mu \leq \max(\mu, \mu') \leq \mu + \mu' \Rightarrow (\mu + \mu ')' \leq \mu ' \Rightarrow (\mu ')' \leq ((\mu + \mu ')')' \)

- \(\mu ' \leq \max(\mu, \mu') \leq \mu + \mu' \Rightarrow (\mu + \mu ')' \leq (\mu ')' \)

then \((\mu + \mu ')' \leq ((\mu + \mu ')')' \). \(\square\)

Notice that no additional hypotheses on the connective +, and the complement ′, are needed for the proof of this theorem: In the algebra of fuzzy sets the excluded-middle principle is a theorem. In conclusion,

In all algebra of fuzzy sets \([0,1]^{X}, \cdot, +, ' \),

- The logic principles of non-contradiction and excluded-middle are theorems, once stated through the concept of self-contradiction.

A very different situation appears if these two principles are stated as it is currently done within logic and classical set theory, that is, by stating

- “P and not P” is false

- “P or not P” is true,

or,
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

- There is no $x$ in $X$ such that “$x$ is $P$ and $x$ is not $P$”
- For all $x$ in $X$ it is “$x$ is $P$ or $x$ is not $P$”

translated into

- $(\mu_P \cdot \mu_{\neg P})(x) = 0$, for all $x$ in $X$
- $(\mu_P + \mu_{\neg P})(x) = 1$, for all $x$ in $X$

that corresponds to ”solve” the equations with fuzzy sets,

$$\mu \cdot \mu' = \mu_0, \ \mu + \mu' = \mu_1,$$

to find for which intersections $\cdot$ and which unions $+$, these equations do hold.

Of course, they do not hold in all cases, for example, with $N = 1-\text{id}$,

- If $\cdot = \text{min}$, it is not always $\min(\mu(x), 1 - \mu(x)) = 0$,
- If $+ = \text{max}$, it is not always $\max(\mu(x), 1 - \mu(x)) = 1$,
- If $\cdot = W$, $(W(a, b) = \max(0, a + b - 1))$ it is $W(a, 1 - a) = \max(0, a + 1 - a - 1) = 0$, and $W(\mu(x), 1 - \mu(x)) = 0$ for all $x$ in $X$
- If $+ = W^*$, $(W^*(a, b) = \min(1, a + b))$, it is $W^*(\mu(x), 1 - \mu(x)) = 1$ for all $x$ in $X$.

That is, there are algebras of fuzzy sets where this forms of non-contradiction or excluded-middle hold, and algebras where this principles do not jointly hold. In the algebras with the triplet $(\text{min}, \text{max}, 1-\text{id})$ do not hold both principles, in the algebras with $(W, \text{max}, 1-\text{id})$ it holds the principle of non-contradiction but not that of excluded-middle, in the algebras with $(\text{min}, W^*, 1-\text{id})$ it holds the excluded-middle but not the principle of non-contradiction, and in the algebras with $(W, W^*, 1-\text{id})$ both principles hold.
Remark 2.2.15. Let us show that with $\mu \in \{0,1\}^X$ it is always $\mu \cdot \mu' = \mu_0$ and $\mu + \mu' = \mu_1$. If $\mu \in \{0,1\}^X$, denote $A = \{x \in X; \mu(x) = 1\}$. Obviously, $\mu = \mu_A$ and $\mu' = \mu_{A^c}$, hence

- $(\mu \cdot \mu') = \mu_A \cdot \mu_{A^c} = \mu_{A \cap A^c} = \mu_0$
- $(\mu + \mu') = \mu_A + \mu_{A^c} = \mu_{A \cup A^c} = \mu_X = \mu_1$

Remark 2.2.16. Results in theorems 2.2.13 and 2.2.14 challenge the usual statement that in fuzzy sets the basic principles of Non-contradiction and Excluded-middle fail. An statement that could conduct to believe that fuzzy set algebras are not properly grounded in a solid logical base.

The fact is, notwithstanding, that these two principles were established before the current ways of considering the problems of logic and, of course, before the nomenclature of set theory. In set theory (or boolean algebras), $A \cap A^c = \emptyset$ and $A \cap A^c \subset (A \cap A^c)^c$ are equivalent formulas since, as it was said, it is

$$B = \emptyset \iff B \subset B^c,$$

an equivalence only verified in the setting of ortholattices (of which boolean algebras are a particular case), but that does not hold on weaker algebraic structures like it is, for example, the case of the above defined algebras of fuzzy sets.

### 2.2.4 Decomposable algebras

**Definition 2.2.17.** An operation with fuzzy sets $* : [0,1]^X \times [0,1]^X \rightarrow [0,1]^X$ is decomposable, or functionally expressible, if it exists a numerical operation $\hat{*} : [0,1] \times [0,1] \rightarrow [0,1]$, such that

$$(\mu * \sigma)(x) = \hat{*}(\mu(x), \sigma(x)),$$

for all $\mu, \sigma$ in $[0,1]^X$ and all $x$ in $X$. Of course, by this formula, a numerical operation $\hat{*}$ allows to define an operation $*$ for fuzzy sets.
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

For example, the operation min in $[0, 1]^X$ is decomposable since, by definition, it is

$$(\min(\mu, \sigma))(x) = \min(\mu(x), \sigma(x))$$

for all $\mu, \sigma$ in $[0, 1]^X$ and all $x$ in $X$.

**Definition 2.2.18.** A function $f : [0, 1]^X \to [0, 1]^X$ is decomposable, or functionally expressible, if it exists a numerical function $\hat{f} : [0, 1] \to [0, 1]$, such that

$$(f(\mu))(x) = \hat{f}(\mu(x)),$$

for all $\mu$ in $[0, 1]^X$ and all $x$ in $X$.

For example, the function $\mu'$ defined by

$$\mu'(x) = 1 - \mu(x)$$

is decomposable because of $N_0 = 1-\text{id}$ gives $\mu'(x) = N_0(\mu(x))$. With $X = [0, 1]$, the function defined by $\mu^x(x) = 1 - \mu(1-x)$ is not decomposable, since if it were such, that is, if there is $N : [0, 1] \to [0, 1]$ such that $1 - \mu(1-x) = N(\mu(x))$, with $\mu \in [0, 1]^X$ such that

- $\mu(0) = 0, \mu(1) = 1 \Rightarrow N(0) = 1 - \mu(1-0) = 1 - \mu(1) = 0, N(1) = 1 - \mu(1-1) = 1 - \mu(0) = 1$
- $\mu(0) = 1, \mu(1) = 0 \Rightarrow N(0) = 1 - \mu(1-0) = 1 - \mu(1) = 1, N(1) = 1 - \mu(1-1) = 1 - \mu(0) = 0$

that is absurd.

The algebras of fuzzy sets $([0, 1]^X, \cdot, +,')$, can be

**Decomposable**, if the three operation $\cdot, +, '$ are decomposable

**Partially decomposable**, if some of the three operations $\cdot, +, '$ is decomposable

**Non decomposable**, if no one of the three operations are decomposable
In what follows we will only deal with decomposable algebras, that is, such that:

There are three functions \( F : [0, 1] \times [0, 1] \to [0, 1], G : [0, 1] \times [0, 1] \to [0, 1], \) and \( N : [0, 1] \to [0, 1] \) with which

\[
(\mu \cdot \sigma)(x) = F(\mu(x), \sigma(x)), (\mu + \sigma)(x) = G(\mu(x), \sigma(x)), \mu'(x) = N(\mu(x)),
\]

for all \( \mu, \sigma \in [0, 1]^X, \) and all \( x \in X. \) For short, \( \mu \cdot \sigma = F \circ (\mu \times \sigma), \mu + \sigma = G \circ (\mu \times \sigma), \mu' = N \circ \mu. \)

In these cases, instead of \([0, 1]^X, \cdot, +, ' \) it is written \(([0, 1]^X, F, G, N)\).

The laws verified by the triplet \((\cdot, +, ')\) force analogous laws for the triplet \((F, G, N)\). For example, linked to the axioms, it is

a) If \( a \leq b, \) then \( F(a, c) \leq F(b, c), F(c, a) \leq F(c, b), \) for all \( c \in [0, 1] \)

b) If \( a \leq b, \) then \( G(a, c) \leq G(b, c), G(c, a) \leq G(c, b), \) for all \( c \in [0, 1] \)

c) If \( a \leq b, \) then \( N(b) \leq N(a) \)

d) \( F(1, a) = F(a, 1) = a, G(0, a) = G(a, 0) = a \)

e) \( N(0) = 1, N(1) = 0, F(0, a) = F(a, 0) = 0, G(1, a) = G(a, 1) = 1 \)

and linked to the theorems 2.2.6 to 2.2.13 it is

- \( F(a, b) \leq \min(a, b) \leq \max(a, b) \leq G(a, b) \) for all \( a, b \) in \([0, 1]\). In particular, it is \( F \leq G. \)

- \( F \) is idempotent \((F(a, a) = a, \) for all \( a \) in \([0, 1]\)), if and only if \( F = \min. \)

- \( G \) is idempotent \((G(a, a) = a, \) for all \( a \) in \([0, 1]\)), if and only if \( G = \max. \)

- It is \( F(a, G(a, b)) = a, \) for all \( a, b \) in \([0, 1]\), if and only if \( F = \min. \)

- It is \( G(a, F(a, b)) = a, \) for all \( a, b \) in \([0, 1]\), if and only if \( G = \max. \)
A triplet \((F, G, N)\) is called dual, or De Morgan triplet, if \(F = N \circ G \circ (N \times N)\), or equivalently, \(G = N \circ F \circ (N \times N)\), that is, \(F(a, b) = N(G(N(a), N(b)))\), or \(G(a, b) = N(F(N(a), N(b)))\), for all \(a, b\) in \([0, 1]\). Notice that, in this case, it is enough to know \(N\) and \(F\) to have \(G\), or \(N\) and \(G\) to have \(F\).

If \(N\) is involutive \((N(N(a)) = a, \text{ for all } a \in [0, 1] \text{ or } N^2 = N)\), the triplet \((\min, \max, N)\) is a dual one.

All triplet \((F, G, N)\) verifies \(F(a, N(a)) \leq G(b, N(b))\), for all \(a, b\) in \([0, 1]\),

- It is \((F(a, N(a)) \leq N(F(a, N(a))), \text{ for all } a \in [0, 1]\)
- It is \(N(F(a, N(a))) \leq N(N(F(a, N(a)))), \text{ for all } a \in [0, 1]\)
- Given \(N\) involutive and \(F\), and denoting by \(G_N\) the dual of \(F\) respect to \(N\), \(G_N(a, b) = N(F(N(a), N(b)))\), it follows \(G_N(N(a), a) \leq N(G_N(N(a), a))\), that with
  \[
  \mu + \sigma = G_N \circ (\mu \times \sigma),
  \]
gives \(\mu' + \mu \leq (\mu' + \mu)'\).

- It is always \(F(a, N(a)) \leq G(b, N(b))\), for all \(a, b\) in \([0, 1]\).

\textbf{Remark 2.2.19.} With classical sets, from \(A \cap B \subset A \cup B\) it results \((A \cup B) \cup (A \cap B) = A \cup B\). In a decomposable theory, to have the analogous law \((\mu + \sigma) + (\mu \cdot \sigma) = \mu + \sigma\) it should be \(G(G(a, b), F(a, b)) = G(a, b)\), for all \(a, b\) in \([0, 1]\), that is verified if, for example, \(G = \max, F = \min\), or \(G = W^*, F = W\).

\textbf{Remark 2.2.20.} Let us show an example of a ‘union’ that is not-decomposable. Define

\[
(\mu + \sigma)(x) = \begin{cases} 
\max(\mu(x), \sigma(x)), & \text{if } \mu \text{ or } \sigma \text{ are in } \{0, 1\}^X \text{ (crisp)} \\
\max(H(\mu), H(\sigma)), & \text{otherwise}.
\end{cases}
\]
It is easy to show that this operation verifies the laws b, d and e in 2.1.2. Hence, it is a union for fuzzy sets that, in addition, is commutative. It is not idempotent, since if \( \mu \in [0, 1]^X - \{0, 1\}^X \), it is \((\mu + \mu)(x) = H(\mu) \neq \mu(x)\). It does not exist a function \( G : [0, 1] \times [0, 1] \to [0, 1] \) such that

\[ (\mu + \sigma)(x) = G(\mu(x), \sigma(x)) \]

for all \( x \in X \) and all \( \mu, \sigma[0, 1]^X \). Indeed, let us suppose that such a \( G \) does exist, and take \( \mu = \mu_{0,5} \). Then

- With \( \sigma = \mu_0 \), it is \((\mu + \sigma)(x) = \max(\frac{1}{2}, 0) = \frac{1}{2} \). Hence \( G(\frac{1}{2}, 0) = \frac{1}{2} \).
- With \( \sigma(x) = x \), is \((\mu + \sigma)(x) = \max(H(\mu), H(\sigma)) = \max(\frac{1}{2}, 1) = 1 \), and \((\mu + \sigma)(0) = 1 = G(\frac{1}{2}, 0)\), that is absurd.

To have a not-decomposable ‘intersection’, it is enough to define, with \( \mu' = 1 - \mu \), the dual operation,

\[
(\mu \cdot \sigma)(x) = \min(\mu(x), \sigma(x)), \quad \text{if } \mu \text{ or } \sigma \text{ are in } \{0, 1\}^X \\
(\mu \cdot \sigma)(x) = \max(H(\mu'), H(\sigma')), \quad \text{otherwise}.
\]

### 2.2.5 Standard algebras of fuzzy sets

An standard algebra of fuzzy sets is a decomposable algebra of fuzzy sets such that:

1. \( \mu \cdot \sigma = \sigma \cdot \mu \), for all \( \mu, \sigma \) in \( [0, 1]^X \) (\( \cdot \) is commutative)
2. \( \mu + \sigma = \sigma + \mu \), for all \( \mu, \sigma \) in \( [0, 1]^X \) (\( + \) is commutative)
3. \( \mu \cdot (\sigma \cdot \lambda) = (\mu \cdot \sigma) \cdot \lambda \), for all \( \mu, \sigma, \lambda \) in \( [0, 1]^X \) (\( \cdot \) is associative)
4. \( \mu + (\sigma + \lambda) = (\mu + \sigma) + \lambda \), for all \( \mu, \sigma, \lambda \) in \( [0, 1]^X \) (\( + \) is associative)
5. \( \mu'' = \mu \), for all \( \mu \) in \( [0, 1]^X \) (\( ' \) is involutive)
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

Hence, writing

\[ \mu \cdot \sigma = T \circ (\mu \times \sigma), \mu + \sigma = S \circ (\mu \times \sigma), \mu' = N \circ \mu, \]

functions \( T, S : [0, 1] \times [0, 1] \rightarrow [0, 1] \) and \( N : [0, 1] \rightarrow [0, 1] \), must verify

- \( T \) is commutative, \( S \) is commutative
- \( T \) is associative, \( S \) is associative
- \( N \) is involutive,

that is:

- \( T(a, b) = T(b, a), S(a, b) = S(b, a) \), for all \( a, b \) in \( [0, 1] \)
- \( T(a, T(b, c)) = T(T(a, b), c), S(a, S(b, c)) = S(S(a, b), c) \), for all \( a, b, c \) in \( [0, 1] \)
- \( N(N(a)) = a \), for all \( a \) in \( [0, 1] \), or \( N \circ N = \text{id} \), or \( N = N^{-1} \).

Functions \( T \) and \( S \) are called t-norms and t-conorms, respectively. Functions \( N \) are strong negations. Hence, \( ([0, 1], T, \leq) \) is an ordered semigroup with neutral 1, and absorbent 0, and \( ([0, 1], S, \leq) \) is also an ordered semigroup but with neutral 0 and absorbent 1. Since \( N(1) = 0 \), it seems that this two kind of ordered semigroups should show some character of duality. This duality goes in the way:

- If \( T \) is a t-norm, \( T_N = N \circ S \circ (N \times N) \) is a t-conorm
- If \( S \) is a t-conorm, \( S_N = N \circ S \circ (N \times N) \) is a t-norm

that are easy to prove. Of course, from 2.1.4,

- If \( T \) is a t-norm, \( T \leq \min \), and \( \min \) is a t-norm
- If \( S \) is a t-conorm, \( \max \leq S \), and \( \max \) is a t-conorm
Hence, for all t-norm $T$ and all t-conorm $S$:

$$T \leq \min \leq \max \leq S,$$

in particular $T \leq S^1$. Even more, the function

$$Z(a, b) = \begin{cases} 
  b, & \text{if } a = 1 \\
  a, & \text{if } b = 1 \\
  0, & \text{otherwise },
\end{cases} = \begin{cases} 
  \min(a, b), & \text{if } a = 1 \text{ or } b = 1 \\
  0, & \text{otherwise },
\end{cases}$$

is obviously a t-norm such that $Z \leq T$ for all t-norm $T$. Consequently,

$$Z^*(a, b) = 1 - Z(1-a, 1-b) = \begin{cases} 
  b, & \text{if } a = 0 \\
  a, & \text{if } b = 0 \\
  1, & \text{otherwise },
\end{cases} = \begin{cases} 
  \max(a, b), & \text{if } a = 0 \text{ or } b = 0 \\
  1, & \text{otherwise },
\end{cases}$$

is a t-conorm such that $S \leq Z^*$ for all t-conorm $S$. Hence, for all t-norm $T$ and all t-conorm $S$,

$$Z \leq T \leq \min \leq \max \leq S \leq Z^*.$$

Notice that $\min(\max)$ is a continuous t-norm (t-conorm), but $Z(Z^*)$ is a discontinuous t-norm (t-conorm). The operations in $[0, 1]$ given by

- $T_{\text{prod}}(a, b) = \text{prod}(a, b) = a \cdot b$
- $T_{W}(a, b) = W(a, b) = \max(0, a + b - 1) = (\max(0, \text{Sum} - 1))(a, b),$

are also continuous t-norms. Then, the dual operations,

- $T_{\text{prod}}^*(a, b) = 1 - T_{\text{prod}}(1-a, 1-b) = 1 - (1-a). (1-b) = a + b - a \cdot b = (\text{Sum} - \text{prod})(a, b)$
- $W^*(a, b) = 1 - W(1-a, 1-b) = \min(1, a + b) = \min(1, \text{Sum})(a, b)$

\textsuperscript{1}Notice that $T \leq S$ mean $T(a, b) \leq S(a, b)$, for all $(a, b) \in [0, 1] \times [0, 1].$
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

are continuous t-conorms. Since it is easy to prove that

\[
W \leq T_{\text{prod}} \leq \min,
\]

it follows \( \max \leq T_{\text{prod}}^* \leq W^* \), and

\[
Z \leq W \leq T_{\text{prod}} \leq \min \leq \max \leq T_{\text{prod}}^* \leq W^*.
\]

Remark 2.2.21. Since \( Z(0.5, 0.5) = 0 \), t-norm \( Z \) has zero-divisors. Analogously, from

\[
W(a, b) = 0 \iff \max(0, a + b - 1) = 0 \iff a + b \leq 1,
\]

it follows that t-norm \( W \) has zero-divisors, for example, \( W(0.5, 0.4) = 0 \); t-norms \( \min \) and \( T_{\text{prod}} \) do not have zero-divisors:

- \( T_{\text{prod}}(a, b) = 0 \iff a = 0 \) or \( b = 0 \)
- \( \min(a, b) = 0 \iff a = 0 \) or \( b = 0 \)

Proposition 2.2.22. The only idempotent t-norm, i.e. \( T(a, a) = a \), for all \( a \in [0, 1] \), is \( T = \min \).

\textbf{Proof.} If \( T \) is idempotent, \( \min(a, b) = T(\min(a, b), \min(a, b)) \leq T(a, b) \)

since \( \min(a, b) \leq a \), and \( \min(a, b) \leq b \). Hence, \( \min(a, b) \leq T(a, b) \leq \min(a, b) \)

implies \( T = \min \). \( \square \)

Proposition 2.2.23. The only idempotent t-conorm, i.e. \( S(a, a) = a \), for all \( a \in [0, 1] \), is \( S = \max \).

\textbf{Proof.} If \( S \) is idempotent, \( \max(a, b) = S(\max(a, b), \max(a, b)) \geq S(a, b) \),

since \( \max(a, b) \geq a \), and \( \max(a, b) \geq b \). Hence, \( \max(a, b) \geq S(a, b) \geq \max(a, b) \)

implies \( S = \max \). \( \square \)

Remark 2.2.24. t-norms can be continuous, like \( \min, T_{\text{prod}}, \) and \( W \), or discontinuous, like \( Z \). They can have zero-divisors, like \( W \) and \( Z \), or not like \( \min \) and \( T_{\text{prod}} \). They can have all elements in \([0, 1]\) idempotent (only \( T = \min \)), only have the idempotents 0 and 1 (like \( T_{\text{prod}} \) and \( W \)), or have some idempotents different from 0 and 1. In any case, since it is always \( T(0, 0) = 0 \) and \( T(1, 1) = 1 \), 0 and 1 are idempotent elements for all t-norms.
Remark 2.2.25. Analogous considerations can be made for t-conorms. There are discontinuous t-conorms like \( Z^* \), and continuous ones like \( T_{\text{prod}}^* \) and \( W^* \). The only for which all elements in \([0,1]\) are idempotent is \( S = \max \). Since \( S(0,0) = 0 \) and \( S(1,1) = 1 \), 0 and 1 are always idempotent, and there are t-conorms that only have these two idempotents (like \( T_{\text{prod}}^* \) and \( W^* \)), as well as those that have some idempotents different from 0,1. There are t-conorms without one-divisors, like max and \( T_{\text{prod}}^* \), and t-conorms with one-divisors like \( W^* \), for example, \( W^*(0.5,0.5) = \min(1,1) = 1 \).

Remark 2.2.26. There is not a characterization theorem for all t-norms (t-conorms), but it is a characterization of the continuous t-norms (t-conorms) that will be presented by means of the following, and easy to prove, results:

- If \( \varphi : [0,1] \to [0,1] \) verifies, 1) If \( x \leq y \), then \( \varphi(x) \leq \varphi(y) \), 2) \( \varphi \) is bijective, 3) \( \varphi(0) = 0, \varphi(1) = 1 \) (\( \varphi \) is an order-automorphism of the ordered interval \([0,1], \leq \)), and \( T \) is a t-norm, then \( T_\varphi = \varphi^{-1} \circ T \circ (\varphi \times \varphi) \) is also a t-norm. Given \( T \), the set \( \{ T_\varphi; \varphi \text{ an order-automorphism} \} \) is called the family of \( T \).

- \( T \) is a continuous t-norm if and only if all t-norms \( T_\varphi \) are continuous.

- If \( S \) is a t-conorm, then \( S_\varphi = \varphi^{-1} \circ S \circ (\varphi \times \varphi) \) is also a t-conorm, and \( S \) is continuous if and only if all t-conorms \( S_\varphi \) are continuous, the set \( \{ S_\varphi; \varphi \text{ an order-automorphism} \} \) is called the family of \( S \).

In particular,

- The family of \( T = \min \) is reduced to the only t-norm min, since \( \varphi^{-1}(\min(\varphi(a),\varphi(b))) = \min(\varphi^{-1}(\varphi(a)),\varphi^{-1}(\varphi(b))) = \min(a,b) \)

- The family of \( T_{\text{prod}} \) contains all continuous t-norms of the form \( \text{prod}_\varphi(a,b) = \varphi^{-1}(\varphi(a) \cdot \varphi(b)) \).

- The family of \( W \) contains all t-norms of the form \( W_\varphi(a,b) = \varphi^{-1}(W(\varphi(a),\varphi(b))) = \varphi^{-1}(\max(0,\varphi(a) + \varphi(b) - 1)) \), and all of them are continuous t-norms.
Notice that no t-norm in the family \( \{ \text{prod}_\varphi \} \) has zero-divisors, since \( \text{prod}_\varphi(a, b) = 0 \iff \varphi(a) \cdot \varphi(b) = 0 \iff \varphi(a) = 0 \) or \( \varphi(b) = 0 \iff a = 0 \), or \( b = 0 \). Instead all t-norms \( W_\varphi \) have zero-divisors, since \( W_\varphi(a, b) = 0 \iff \max(0, \varphi(a) + \varphi(b) - 1) \iff \varphi(a) + \varphi(b) \leq 1 \). Of course, neither t-norms \( \text{prod}_\varphi \), nor \( W_\varphi \), have more idempotents than 0, and 1:

- \( a = W_\varphi(a, a) = \varphi^{-1}(\max(0, 2\varphi(a) - 1)) \iff \varphi(a) = \max((0, 2\varphi(a) - 1) \iff \varphi(a) = 0 \) or \( \varphi(a) = 1 \) or \( a = 0 \) or \( a = 1 \).
- \( a = \text{prod}_\varphi(a, a) = \varphi^{-1}(\varphi(a), \varphi(a)) \iff \varphi(a) = \varphi(a), \varphi(a) \iff \varphi(a) = 0 \) or \( \varphi(a) = 1 \) or \( a = 0 \) or \( a = 1 \).

Analogously,

- The family of \( S = \max \), only contains this t-conorm.
- The family of \( T^*_\text{prod} \) contains all t-conorms of the form
  \[ \text{prod}^*_\varphi(a, a) = \varphi^{-1}(\text{prod}^*\varphi(a), \varphi(b))) = \varphi^{-1}(\varphi(a) + \varphi(b) - \varphi(a) \cdot \varphi(b)) \]
- The family of \( W^* \) contains all t-conorms of the form
  \[ W^*(a, a) = \varphi^{-1}(W^*(\varphi(a), \varphi(b))) = \varphi^{-1}(\min(1, \varphi(a) + \varphi(b))) \]

Remark 2.2.27. The order-automorphism \( \varphi \) plays the role of a functional parameter. By taking, \( \varphi(x) = x^r \), it follows, for example,

\[ W_\varphi(a, b) = \sqrt[\varphi]{\max(0, a^r + b^r - 1)}, W^*_\varphi(a, b) = \sqrt[\varphi]{\min(1, a^r + b^r)} \]

giving a family of t-norms (t-conorms) depending on the numerical parameter \( r > 0 \). Notice that with \( \varphi(x) = x^r \),

\[ \text{Prod}_\varphi(a, b) = \sqrt[\varphi]{a^r \cdot b^r} = a \cdot b = \text{Prod}(a, b), \]

but

\[ \text{Prod}^*_\varphi(a, b) = \varphi^{-1}(\varphi(a) + \varphi(b) - \varphi(a) \cdot \varphi(b)) = \sqrt[\varphi]{a^r + b^r - a^r \cdot b^r}. \]
2.2.6 Strong negations

As it was said before, a strong negation is a function $N : [0, 1] \rightarrow [0, 1]$ such that

- $N(0) = 1$
- If $a \leq b$, then $N(b) \leq N(a)$
- $N(N(a)) = a$, for all $a \in [0, 1]$, or $N^2 = id$. 

Notice that $N^2 = id$ is equivalent to $N = N^{-1}$, that shows $N$ is a continuous function: It is $N(1) = N(N(0)) = 0$, and if $a < b$ it should be $N(b) < N(a)$ since $N(b) = N(a)$ would imply $N(N(b))N(N(a))$, or $a = b$. Hence, $N$ is strictly decreasing.

Since $N$ is continuous, the equation $N(x) = x$ has solutions, but there is only one. Suppose $N(x_1) = x_1$ and $N(x_2) = x_2$. Either $x_1 \leq x_2$, or $x_2 < x_1$. In the first case, it follows $N(x_2) \leq N(x_1)$, or $x_2 \leq x_1$, and $x_1 = x_2$. In the second case, $N(x_1) < N(x_2)$, or $x_1 < x_2$, that is absurd. Then, each strong negation has a single fixed point $N(n) = n$, in the open interval $(0, 1)$, since $N(0) = 1, N(1) = 0$ shows that 0 and 1 are not fixed points.

**Remark 2.2.28.** In the classical case (a boolean algebra $L$, or a power set $P(X)$), the transformation

$$F : P(X) \rightarrow P(X), F(A) = A^c,$$

has no fixed points, since $A = A^c$ implies $A \cap A = A \cap A^c$, or $A = \emptyset$, and $\emptyset^c = X$. Nevertheless, with fuzzy sets

$$F : [0, 1]^X \rightarrow [0, 1]^X, F(\mu) = \mu' = N \circ \mu,$$

the equation $\mu = \mu'$, $N(\mu(x)) = \mu(x)$ for all $x$ in $X$, has the only solution $\mu(x) = n$ for all $x$ in $X$, that is $\mu = \mu_n$.

In the fuzzy case, that mapping $F$ shows a kind of symmetry that is not in the crisp case.
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

An order-automorphism of the ordered unit interval $([0, 1], \leq) \varphi : [0, 1] \to [0, 1]$, verifies,

- $\varphi(0) = 0, \varphi(1) = 1$
- If $a < b$, then $\varphi(a) < \varphi(b)$.

Hence, $\varphi$ is continuous, and its inverse function $\varphi^{-1}$ verifies,

- $\varphi^{-1}(0) = 0, \varphi^{-1}(1) = 1$
- If $a < b$, then $\varphi^{-1}(a) < \varphi^{-1}(b)$.

Let us denote by $N_\varphi$ the function $N_\varphi : [0, 1] \to [0, 1]$ defined by

$$N_\varphi(a) = \varphi^{-1}(1 - \varphi(a)), \text{ for all } a \in [0, 1]$$

**Proposition 2.2.29.** $N_\varphi$ is a strong negation.

**Proof.** It is $N_\varphi(0) = \varphi^{-1}(1 - \varphi(0)) = \varphi^{-1}(1) = 1$. In addition, if $a \leq b$, it follows $1 - \varphi(b) \leq 1 - \varphi(a)$, and $\varphi^{-1}(1 - \varphi(b)) \leq \varphi^{-1}(1 - \varphi(a))$, or $N_\varphi(b) \leq N_\varphi(a)$. Finally,

$$N_\varphi(N_\varphi(a)) = N_\varphi(\varphi^{-1}(1 - \varphi(a))) = \varphi^{-1}(1 - \varphi(\varphi^{-1}(1 - \varphi(a)))) =$$

$$\varphi^{-1}(1 - 1 + \varphi(a)) = \varphi^{-1}(\varphi(a)) = a$$

$\square$

**Theorem 2.2.30.** If $N$ is a strong negation, there are order-automorphisms $\varphi$ such that $N = N_\varphi$.

**Proof.** Let it be $n = N(n) \in (0, 1)$ the fixed point of $N$, and consider an strictly non-decreasing function $h : [0, n] \to [0, \frac{1}{2}]$ such that $h(0) = 0$ and $h(n) = \frac{1}{2}$. With $h$ define the function $\varphi : [0, 1] \to [0, 1]$ by

$$\varphi(x) = \begin{cases} 
  h(x), & \text{if } x \in [0, n] \\
  1 - h(N(x)), & \text{if } x \in (n, 1].
\end{cases}$$
This function $\varphi$ is, obviously, continuous, strictly increasing\(^2\), and verifies $\varphi(0) = h(0) = 0, \varphi(1) = 1 - h(N(1)) = 1 - h(0) = 1$. Then

- If $x \in [0, n)$, or $N(x) \in (n, 1]$, $\varphi(N(x)) = 1 - h(x) = 1 - \varphi(x)$, and $N(x) = \varphi^{-1}(1 - \varphi(x))$.
- If $x = n$, $N(n) = n = h^{-1}(\frac{1}{2}) = \varphi^{-1}(\frac{1}{2})$, or $N(n) = \varphi^{-1}(1 - \varphi(x))$
- If $x \in (n, 1]$, or $N(x) \in [0, n)$, $\varphi(N(x)) + \varphi(x) = h(N(x)) + 1 - h(N(x)) = 1$

In conclusion, $N(x) = \varphi^{-1}(1 - \varphi(x))$, for all $x$ in $[0, 1]$, or $N = N_{\varphi}$.

Notice that the proof of last theorem shows clearly that the order-automorphism $\varphi$ such that $N = N_{\varphi}$ is not unique. Notice also that with $\varphi = \text{id}_{[0, 1]}$ it follows $N(x) = 1 - x$, the fundamental strong negation, with which it results $N = N_{\varphi} = \varphi^{-1} \circ (1 - \text{id}_{[0, 1]}) \circ \varphi = \varphi^{-1} \circ N \circ \varphi$, that is, all strong negations belong to the family of $N_0(x) = 1 - x$. Nevertheless, in all cases it is $n = \varphi^{-1}(\frac{1}{2})$.

If $\varphi(x) = x^2$, it results $N_{\varphi}(x) = \sqrt{1 - x^2}$, called the circular negation. If $\varphi(x) = \frac{2x}{1+x}$, or $\varphi^{-1}(x) = \frac{x}{2-x}$, it follows $N_{\varphi}(x) = \varphi^{-1}(1 - \frac{2x}{1+x}) = \varphi^{-1}(\frac{x}{1+x}) = \frac{1-x}{1+x}$, that is the strong negation $N_3$ of the before mentioned Sugeno’s negations. With $\varphi(x) = \frac{1}{\lambda} \ln(1 + \lambda x^\alpha), \lambda > -1, \alpha > 0$, it follows the bi-parametric family $N_{\varphi}(x) = (\frac{1-x^\alpha}{1+\lambda x^\alpha})^{\frac{1}{\lambda}}$. With $\alpha = 1$ it is obtained the Sugeno’s family of strong negations $N_\lambda = \frac{1-x}{1+\lambda x}$ (\lambda-1) that only depends on one single parameter.

**Remark 2.2.31.** The only linear strong negation $N$ is $N = N_0$, since from $N(a) = \alpha a + \beta$, with $N(0) = 1 = \beta$ and $N(1) = 0 = \alpha + 1$, follows $\alpha = -1$ and $N(a) = 1 - a$.

**Remark 2.2.32.** The only “rational” strong negations $N$ of the form $N(x) = \frac{ax+b}{cx+d}$, $a, b$ different of 0, are those $N_\lambda(\lambda > 1)$ in the Sugeno’s family. It follows from:

\(^2\text{For } (x < y) \text{ is evident that } \varphi(x) < \varphi(y) \text{ if either } x, y \in [0, n], \text{ or } x, y \in (n, 1]. \text{ If } x \in [0, n], y \in (n, 1] \text{ and } x < y, \text{ since } h(x) + h(N(x)) < 1, \text{ it is } h(x) < 1 - h(N(x)), \text{ or } \varphi(x) < \varphi(y)\)
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

- \( N(0) = 1 = \frac{b}{d}, \) or \( d = b \)
- \( N(1) = 0 = \frac{a+b}{c+d}, \) or \( a = -b \)

that gives

\[
N(x) = \frac{-bx + b}{cx + b} = \frac{b(1 - x)}{cx + b} = \frac{1 - x}{1 + \frac{c}{b}x}.
\]

To have \( 0 \leq N(x) \leq 1, \) it should be \( 1 - x \leq 1 + \frac{c}{b}. \) But \( -1 = \frac{c}{b} \) implies \( N(x) = 1, \) that is not an strong negation. Hence it is \( -1 < \frac{c}{b}, \) and with \( \lambda = \frac{c}{b}, \) it follows \( N(x) = \frac{1-x}{1+\frac{c}{b}x} = N_\lambda(x), \) with \(-1 < \lambda.\)

### 2.2.7 Continuous t-norms and t-conorms

As it was said, the only t-norm that is idempotent for all \( a \) in \([0, 1], \) is \( T = \min, \) and the t-norms in \( \{prod\} \cup \{W\} \) only have the idempotents 0 and 1. As it was also said, there are t-norms with several (but not all) idempotent elements. For example, the function

\[
T(x, y) = \begin{cases} 
\frac{1}{3} + \frac{1}{3}W(3x - 1, 3y - 1), & \text{if } (x, y) \in \left[\frac{1}{3}, \frac{2}{3}\right]^2 \\
\min(x, y), & \text{otherwise},
\end{cases}
\]

that as it is easy to prove is a t-norm, verifies

- \( T(x, x) = \min(x, x) = x, \) if \( x \notin \left[\frac{1}{3}, \frac{2}{3}\right] \)
- \( T\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} + \frac{1}{3}W(0, 0) = \frac{1}{3} + \frac{1}{3} \cdot 0 = \frac{1}{3} \)
that is, all elements in \([0, 1] - [\frac{1}{3}, \frac{2}{3}]\), as well \(\frac{1}{3}\) and \(\frac{2}{3}\) are idempotent for \(T\), and the elements in \((\frac{1}{3}, \frac{2}{3})\) are not idempotent.

Look that an analogous result is obtained when changing \(W\) by \(\text{prod}\) in the above expression of \(T\). Without proof it follows the theorem that completely characterizes all continuous t-norms.

**Theorem 2.2.33.** \(T\) is a continuous t-norm if and only if,

1. \(T = \text{min}\), \(T\) is in the family of \(\text{min}\)

2. \(T = \text{prod}_\varphi\), \(T\) is in the family of \(\text{prod}\)

3. \(T = W_\varphi\), \(T\) is in the family of \(W\)

4. There exist an index set (finite or countable infinite), a family of pairwise disjoint open intervals of \([0, 1]\), \([a_i, b_i]; i \in I\), and a family of t-norms \(T_i \in \{\text{prod}\} \cup \{W\} (i \in I)\), such that

\[
T(x, y) = \begin{cases} 
    a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & \text{if } (x, y) \in [a_i, b_i]^2 \\
    \min(x, y), & \text{otherwise},
\end{cases}
\]

for any \(x, y\) in \([0, 1]\).

The continuous t-norm of the fourth type are called ordinal-sums of the continuous t-norms \(T_i \in \{\text{prod}\} \cup \{W\}\).

**Remark 2.2.34.** Why the names t-norm and t-conorm? The “t” comes from “triangular”, because these functions were introduced to formalize the triangular property of probabilistic distances, i.e. distances whose values are something like the probability that the numerical distance between two points
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

is less than a given number. They were introduced by Karl Menger with the name triangular-norms without considering associativity.

The name of t-conorm refers to the duality with a t-norm, since $S$ is a t-conorm if and only if $1 - S(1 - x, 1 - y)$ is a t-norm. In general, it should be pointed out that, for each strong negation $N$, $S$ is a t-conorm if and only if $N \circ S \circ (N \times N)$ is a t-norm.

**Theorem 2.2.35.** $S$ is a continuous t-conorm if and only if,

1. $S = \max$, $S$ is in the family of $\max$
2. $S = \prod^*$, $S$ is in the family of $\prod^*$
3. $S = W^*$, $S$ is in the family of $W^*$
4. There exist an index set (finite or countable infinite), a family of pairwise disjoint open intervals of $[0, 1]$, $\{(a_i, b_i); i \in I\}$, and a family of t-conorms $S_i \in \{\prod^*\} \cup \{W^*\} (i \in I)$, such that

$$S(x, y) = \begin{cases} a_i + (b_i - a_i)S_i \left( \frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right), & \text{if } (x, y) \in [a_i, b_i]^2 \\ \max(x, y), & \text{otherwise,} \end{cases}$$

for any $x, y$ in $[0, 1]$.

**Remark 2.2.36.** Notice that with both ordinal-sums of t-norms and of t-conorms, provided it is $[0, 1] = [0, b_1] \cup [b_1, b_2] \ldots \cup [b_{n-1}, b_n] \cup [b_n, 1]$, a finite partition of the unit interval $[0, 1]$, like, for example

![Diagram](attachment:image.png)
the only idempotent elements are \( b_1, b_2, b_3, b_4 \), etc, as well as 0 and 1, that is, the points giving the partition of \([0, 1]\).

Remark 2.2.37. Although currently only continuous t-norms in \( \{\min\} \cup \{\text{prod}\} \cup \{W\} \) are taken into account in both theoretic fuzzy logic and its applications, it should be pointed out that provided there are, at least, two statements ‘\( x \) is \( P \)’ and ‘\( x \) is \( Q \)’ such that \( \mu_P(x), \mu_Q(x) \not\in \{0, 1\} \) and \( \mu_P \land \mu_Q = \mu_P, \mu_Q \land \mu_Q = \mu_Q \), the only possibility for representing \( \mu \cdot \sigma = T \circ (\mu \times \sigma) \), is by taking as continuous t-norm \( T \) an ordinal-sum with the single interval \( (\min(\mu_P(x), \mu_Q(x)), \max(\mu_P(x), \mu_Q(x))) \).

Remark 2.2.38. Which t-norms are strictly non-decreasing in the sense that if \( 0 < a < b < 1 \), then \( T(a, c) < T(b, c) \) for all \( c \in [0, 1] \)?

- If \( T = \min \), the answer is negative. For example, \( 0.3 < 0.5 \), but \( \min(0.2, 0.3) = \min(0.2, 0.5) = 0.2 \)
- If \( T = W_\varphi \), the answer is also negative. For example, \( 0.3 < 0.5 \), but \( W(0.2, 0.3) = W(0.2, 0.5) = 0 \)
- If \( T = \text{prod}_\varphi \), the answer is positive, since: \( a < b \Rightarrow \varphi(a) < \varphi(b) \Rightarrow \varphi(a) \cdot \varphi(c) < \varphi(b) \cdot \varphi(c) \Rightarrow \varphi^{-1}(\varphi(a) \cdot \varphi(c))) < \varphi^{-1}(\varphi(b) \cdot \varphi(c))) \), or \( \text{prod}_\varphi(a, c) < \text{prod}_\varphi(b, c) \), because \( \varphi(c) \in (0, 1] \).
- If \( T \) is an ordinal-sum, it can’t be strictly non-decreasing because of the values it takes with \( \min \).

Analogously, the only t-conorms that are strictly non-decreasing are those in \( \{\text{prod}_\varphi^*\} \)

2.2.8 Laws of fuzzy sets

As it was said, in all standard algebras \( ([0, 1]^X, T, S, N) \) of fuzzy sets the triplet \( (T, S, N) \) share the following common properties:

1. \( T \) and \( S \) are commutative and associative
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

2. 1 is neutral for $T$, and 0 is neutral for $S$

3. 0 is absorbent for $T$, and 1 is absorbent for $S$

4. For all $T$ and $S$, it is $T \leq \min < \max \leq S$

5. Each $T$ ($S$) is non decreasing in the two variables

6. $N$ is involutive, strictly decreasing and such that $N(0) = 1$,


\begin{itemize}
\item A list of properties that gives some basic laws for fuzzy sets in the standard algebras, like
\begin{itemize}
\item $\mu \cdot \sigma = \sigma \cdot \mu, \mu + \sigma = \sigma + \mu,$
\item $\mu + (\sigma + \lambda) = (\mu + \sigma) + \lambda = (\sigma + \mu) + \lambda = \lambda + (\sigma + \mu)$
\item $\mu \cdot \mu_1 = \mu, \mu + \mu_0 = \mu, \mu + \mu_1 = \mu_1, \mu \cdot \mu_0 = \mu_0$
\item If $\mu \leq \sigma$, then $\mu \cdot \lambda \leq \sigma \cdot \lambda$, and $\lambda + \mu \leq \sigma + \lambda$
\end{itemize}
\end{itemize}

\begin{itemize}
\item etc.
\end{itemize}

Anyway, a lot of laws typical of classical sets are not always valid in all standard algebras of fuzzy sets. For example, $(\mathcal{P}(X), \cap, \cup, ^c)$ is a boolean algebra and no one $([0,1]^X, T, S, N)$ is a boolean algebra. In particular, $(\mathcal{P}(X), \cap, \cup)$ is a lattice and the only standard algebra that is a lattice is that with $T = \min$ and $S = \max$. Let us study in which standard algebras some common laws of crisp sets do hold.

\subsection*{2.2.8.1 Distributive laws}

With classical sets it always do hold the two distributive laws

\begin{itemize}
\item 1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
\item 2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
\end{itemize}
and the question is for which triplets \((T, S, N)\) do hold the corresponding laws with fuzzy sets

1. \(\mu \cdot (\sigma + \lambda) = \mu \cdot \sigma + \mu \cdot \lambda\),

2. \(\mu + (\sigma \cdot \lambda) = (\mu + \sigma) \cdot (\mu + \lambda)\).

This questions correspond to solve the functional equations in the unknowns \(T\) and \(S\):

\[
\begin{align*}
T(a, S(b, c)) &= S(T(a, b), T(a, c)) & (2.3) \\
S(a, T(b, c)) &= T(S(a, b), S(a, c)) & (2.4)
\end{align*}
\]

for all \(a, b, c\) in \([0, 1]\).

**Lemma 2.2.39.** Equation (2.3) does hold if and only if \(S = \text{max}\).

**Proof.** With \(b = c = 1\), is \(T(a, S(1, 1)) = S(T(a, 1), T(a, 1))\) or \(a = S(a, a)\). That is \(S = \text{max}\).

Provided \(S = \text{max}\), the equation is \(T(a, \max(b, c)) = \max(T(a, b), T(a, c))\). If either \(b \leq c\) or \(c \leq b\), it is immediate to check its validity for all t-norm \(T\). □

**Lemma 2.2.40.** Equation (2.4) does hold if and only if \(T = \text{min}\).

**Proof.** With \(b = c = 0\), is \(S(a, 0) = a = T(a, a)\). That is \(T = \text{min}\).

Provided \(T = \text{min}\), the equation is \(S(a, \min(b, c)) = \min(S(a, b), S(a, c))\). If either \(b \leq c\) or \(c \leq b\), it is immediate to check its validity for all t-conorm \(T\). □

Hence,

- In all standard algebras with \((T, \text{max})\), it holds \(\mu \cdot (\sigma + \lambda) = \mu \cdot \sigma + \mu \cdot \lambda\),

- In all standard algebras with \((\text{min}, S)\), it holds \(\mu + (\sigma \cdot \lambda) = (\mu + \sigma) \cdot (\mu + \lambda)\)

- The two distributive laws (2.3) and (2.4) do jointly hold if and only if \(T = \text{min}\) and \(S = \text{max}\).
2.2.8.2 De Morgan laws

With classical sets it always do hold the De Morgan, or duality, laws

1. \( A \cup B = (A^c \cap B^c)^c \)
2. \( A \cap B = (A^c \cup B^c)^c \)

showing that one of the two operators \( \cap, \cup \) can be defined by the other and the complementation. With fuzzy sets these laws are

1. \( \mu \cdot \sigma = (\mu' + \sigma')', \text{ or } (\mu \cdot \sigma)' = \mu' + \sigma' \)
2. \( \mu + \sigma = (\mu' \cdot \sigma')', \text{ or } (\mu + \sigma)' = \mu' \cdot \sigma' \),

that correspond to the functional equations in the unknowns \( T, S \) and \( N \)

- \( T(a, b) = N(S(N(a), N(b))) \)
- \( S(a, b) = N(T(N(a), T(b))) \)

for all \( a, b \) in \([0, 1]\).

Obviously, law (1) does hold if and only if \( T = N \circ S \circ (N \times N) \) and law (2) does hold if and only if \( S = N \circ T \circ (N \times N) \), two formulas that are equivalent since, for example, from the first it follows (with \( N^2 = \text{id} \)) \( N \circ T = S \circ (N \times N) \) or \( N \circ S = T \circ (N \times N) \), that is, the second formula.

Hence, the two De Morgan laws hold in an algebra given by the triplet \((T, S, N)\) if and only if \( T = N \circ S \circ (N \times N) \), that is, \( T \) and \( S \) are \( N \)-dual.

2.2.8.3 Non-contradiction principle \( \mu \mu' = \mu_0 \).

With classical sets it always holds \( A \cap A^c = \emptyset \). With fuzzy sets, when is it

\[ \mu \cdot \mu' = \mu_0? \]

The equation to be solved is

\[ T(a, N(a)) = 0, \text{ for all } a \in [0, 1] \]

in the unknowns \( T \) and \( N \).
CHAPTER 2. ALGEBRAS OF FUZZY SETS

Theorem 2.2.41. If $T$ is a continuous t-norm, and $N$ is a strong negation, it is $T(a, N(a)) = 0$ for all $a \in [0, 1]$, if and only if $T = W_\varphi$ and $N \leq N_\varphi$.

Proof. With the fixed point $n \in (0, 1)$ of $N$, it follows $T(n, n) = 0$, that is, $T$ has zero-divisors. Hence, $T = W_\varphi$, and $W_\varphi(a, N(a)) = \varphi^{-1}(\max(0, \varphi(a) + \varphi(N(a)) - 1)) = 0$, or $\max(0, \varphi(a) + \varphi(N(a)) - 1) = 0$, or $\varphi(a) + \varphi(N(a)) - 1 \leq 0$, that implies $\varphi(N(a)) \leq 1 - \varphi(a)$, or $N(a) \leq \varphi^{-1}(1 - \varphi(a)) = N_\varphi(a)$, for all $a \in [0, 1]$. Hence $N \leq N_\varphi$. The reciprocal is a simple calculation. \qed

Then, the non-contradiction principle $\mu \cdot \mu' = \mu_0$ holds if and only if $T = W_\varphi$ and $N \leq N_\varphi$, for any order-automorphism $\varphi$ and any t-conorm $S$. For example, it holds (with $\varphi = \text{id}$) if $T = W, S = \text{max}, N = N_0$, and it does not hold provided $T = \min, \text{or } S = \text{prod}_\varphi$.

2.2.8.4 Excluded-middle principle $\mu + \mu' = \mu_1$.

With classical sets it always holds $A \cup A^c = X$. When is it $\mu + \mu' = \mu_1$? When it does hold the equation $S(a, N(a)) = 1$ for all $a \in [0, 1]$?

Theorem 2.2.42. If $S$ is a continuous t-conorm, and $N$ is a strong negation, it is $S(a, N(a)) = 1$ for all $a \in [0, 1]$, if and only if $S = W_\psi$ and $N_\psi \leq N$.

Proof. With $N(n) = n \in (0, 1)$, it follows $S(n, n) = 1$. That is $S = W_\psi^*$, and $1 = W_\psi^*(a, n(a)) = \psi^{-1}(\min(1, \psi(a) + \psi(N(a))))$, or $1 = \min(1, \psi(a) + \psi(N(a)))$. Hence, $1 \leq \psi(a) + \psi(N(a))$, or $N_\psi(a) = \psi^{-1}(1 - \psi(a) \leq N(a))$. That is, $N_\psi \leq N$. The reciprocal is a simple calculation. \qed

Then, the excluded-middle principle $\mu + \mu' = \mu_1$ holds if and only if $S = W_\psi^*$ and $N_\psi \leq N$, for any order automorphism $\psi$ and any t-norm $T$. For example, it holds (with $\psi = \text{id}$) if $S = W^*, T = \min, N = N_0$, but it does not hold provided $S = \text{max}$ or $S = \text{prod}_\psi$.

2.2.8.5 Both principles of Non-contradiction and Excluded-middle

From last theorems it immediately follows that,
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

Theorem 2.2.43. In a standard algebra of fuzzy sets with a triplet \((T, S, N)\), it holds \(\mu \cdot \mu' = \mu_0\) and \(\mu + \mu' = \mu_1\) if and only if \(T = W_\varphi\), \(S = W_\psi^*\), and \(N_\psi \leq N \leq N_\varphi\).

In particular, they hold if \(T = W\), \(S = W^*\), and \(N = N_0\), or with \(\varphi(x) = x^2\) and \(\psi(x) = x\), they hold with the triplet:

\[
T(x, y) = \sqrt{\max(0, x^2 + y^2 - 1)},
S(x, y) = \min(1, x + y),
1 - x \leq N(x) \leq \sqrt{1 - x^2},
\]

for all \(x, y\) in \([0, 1]\).

Of course, with \(\varphi = \psi\), the principles hold with \(W_\varphi\), \(W_\psi^*\), and \(N = N_\varphi\).

2.2.8.6 Laws of absorption

With classical sets, the absorption laws \(A \cap (A \cup B) = A\), and \(A \cup (A \cap B) = A\), always do hold. With fuzzy sets, the formulas and respective equations

- \(\mu \cdot (\mu + \sigma) = \mu, \quad T(a, S(a, b)) = a\)
- \(\mu + (\mu \cdot \sigma) = \mu, \quad S(a, T(a, b)) = a\)

must be studied to find for which algebras these laws do hold.

Lemma 2.2.44. If \(T\) and \(S\) are, respectively, a continuous t-norm and a t-conorm, it is \(T(a, S(a, b)) = a\) for all \(a, b\) in \([0, 1]\) if and only if \(T = \min\).

Proof. If \(T = \min\), since \(a \leq S(a, b)\), it follows \(\min(a, S(a, b)) = a\). With \(b = 0\), the equation gives \(T(a, a) = a\), and \(= \min\). ☐

Lemma 2.2.45. If \(T\) and \(S\) are, respectively, a continuous t-norm and a continuous t-conorm, it is \(S(a, T(a, b)) = a\) for all \(a, b\) in \([0, 1]\) if and only if \(S = \max\).
Proof. Since $T(a, b) \leq a$, it follows $\max(a, T(a, b)) = a$. With $b = 1$, the equation gives $S(a, a) = a$, and $= \max$. □

Hence,

- The law $\mu \cdot (\mu + \sigma) = \mu$, holds for all $S$ and $T = \min$
- The law $\mu + (\mu \cdot \sigma) = \mu$, holds for all $T$ and $S = \max$
- The two laws hold jointly if and only if $T = \min$ and $S = \max$.

2.2.8.7 The law of von Neumann

With classical sets it always holds the law of von Neumann, or law of the perfect repartition,

$$A = (A \cap B) \cup (A \cap B^c),$$

that follows from $A = A \cap X = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$, and generalizes that of the excluded-middle since $A = X$ implies $X = (B \cap X) \cup (X \cap B^c) = B \cup B^c$.

From that law, by duality it follows $A^c = (A \cap B)^c \cap (A \cap B^c)^c = (A^c \cup B^c) \cap (A^c \cup B)$, that is

$$A = (A \cup B) \cap (A \cup B^c),$$

a law that generalizes that of non-contradiction since $A = \emptyset$ implies $\emptyset = B \cap B^c$.

With fuzzy sets, the question is the validity of the laws

$$\mu = \mu \cdot \sigma + \mu \cdot \sigma', \mu = (\mu + \sigma) \cdot (\mu + \sigma'),$$

or of the functional equations

$$a = S(T(a, b), T(a, N(b))), \ a = T(S(a, b), S(a, N(b)))$$
Lemma 2.2.46. The equation $a = S(T(a, b), T(a, N(b)))$ holds if and only if $T = prod_\varphi, S = W^*_\varphi, N = N_\varphi$.

Proof. If $T = prod_\varphi, S = W^*_\varphi, N = N_\varphi$, it is $S(T(a, b), T(a, N(b))) = W^*_\varphi(prod_\varphi(a, b), prod_\varphi(a, N_\varphi(b))) = \varphi^{-1}(W^*(\varphi(prod_\varphi(a, b)), \varphi(prod_\varphi(a, N_\varphi(b)))) = \varphi^{-1}(\varphi(a) \cdot \varphi(b), \varphi(a) \cdot \varphi(N_\varphi(b))) = \varphi^{-1}(\min(1, \varphi(a) \cdot \varphi(b) + \varphi(a) \cdot (1 - \varphi(b)) = \varphi^{-1}(\min(1, \varphi(a) \cdot \varphi(b) + \varphi(a) - \varphi(a) \cdot \varphi(b))) = \varphi^{-1}(\min(1, \varphi(a)) = a$.

The proof of the reciprocal will be avoided since it is technically complex. Let us say only that $a = S(T(a, b), T(a, N(b)))$ gives, with $a = 1, 1 = S(b, N(b))$, that implies $S = W^*$ and $N_\varphi \leq N$. □

It can be also proven that $a = T(S(a, b), S(a, N(b)))$ if and only if $T = W_\varphi, S = prod_\varphi, N = N_\varphi$. Notice only that $a = 0$ gives $T(b, N(b)) = 0$, or $T = W_\varphi$ and $N = N_\varphi$.

Notice that the verification of von Neumann’s laws require, in the case of fuzzy sets, non-dual theories.

2.2.8.8 Which standard algebra is closer to a boolean algebra?

The results in last section can be summarized in the following table.

Hence, the algebras with the triplets $(\min, \max, N)$ are the ones that preserve more structural boolean properties. Indeed, these algebras preserve all the basic boolean laws except those of non-contradiction and excluded-middle. They are distributive pseudo-complemented lattices, that is, De Morgan algebras that, in addition and like all algebras of fuzzy sets, verify the law of Kleene,

$$T(a, N(a)) \leq S(b, N(b)),$$

for all $a, b$ in $[0, 1]$. The algebras given by the triplets $(\min, \max, N)$ are De Morgan-Kleene algebras.
2.2.8.9 Last comments

It can be considered, in addition to the structural boolean laws, the cases that can be derived from them, for example,

\[(A \cap B^c)^c = B \cup (A^c \cap B^c),\]

that follows from \(B \cup (A^c \cap B^c) = (B \cup A^c) \cap (B \cup B^c) = (B \cup A^c) \cap X = B \cup A^c = (A \cap B^c)^c\). This law, in fuzzy set theory, is translated by

\[(\mu \cdot \sigma')' = \sigma + \mu' \cdot \sigma'\]

or

\[N(T(a, N(b))) = S(b, T(N(a), N(b))),\]

that holds with \(T = prod_\varphi, S = W_\varphi^*, N = N_\varphi\).

Nevertheless, not all derived law has solution within the standard algebras of fuzzy sets, as it is the case with \((A \cup A) \cap (A \cap A^c) = \emptyset\) (or \(A \cap \emptyset = \emptyset\)).

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>S</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice</td>
<td>min</td>
<td>max</td>
<td>all</td>
</tr>
<tr>
<td>Identity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T(a, 1) = a, T(a, 0) = 0)</td>
<td>all</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(S(a, 0) = a, S(a, 1) = 1)</td>
<td>-</td>
<td>all</td>
<td>-</td>
</tr>
<tr>
<td>Commutativity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T(a, b) = T(b, a))</td>
<td>all</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(S(a, b) = S(b, a))</td>
<td>-</td>
<td>all</td>
<td>-</td>
</tr>
<tr>
<td>Associativity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T(a, T(b, c)) = T(T(a, b), c))</td>
<td>all</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(S(a, S(b, c)) = S(S(a, b), c))</td>
<td>-</td>
<td>all</td>
<td>-</td>
</tr>
<tr>
<td>Involution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(N(N(a)) = a)</td>
<td>-</td>
<td>-</td>
<td>all</td>
</tr>
</tbody>
</table>
2.2. THE CONCEPT OF AN ‘ALGEBRA OF FUZZY SETS’.

| B1. Idempotency |
|-----------------|-----------------|-------------|
| $T(a,a) = a$    | Min             | -           |
| $S(a,a) = a$    | -               | Max         |

| Distributivity |
|----------------|-----------------|-------------|
| $T(a, S(b,c)) = S(T(a,b), T(a,c))$ | all | Max |
| $S(a, T(b,c)) = T(S(a,b), S(a,c))$ | Min | all |

| Absorption |
|------------|-----------------|-------------|
| $T(a, S(a,b)) = a$ | Min | all |
| $S(a, T(a,b)) = a$ | all | Max |

| Non-Contradiction |
|-------------------|-----------------|-------------|
| $T(a, N(a)) = 0$  | $W_\varphi$     | $N \leq N_\varphi$ |

| Excluded-middle |
|-----------------|-----------------|-------------|
| $S(a, N(a)) = 1$| -               | $W_\varphi^*$ $N \geq N_\varphi$ |

| De Morgan’s laws |
|------------------|------------------|-------------|
| $N(T(a,b)) = S(N(a), N(b))$ | $T = N \circ S \circ N \times N$ |
| $N(S(a,b)) = T(N(a), N(b))$ | $T = N \circ S \circ N \times N$ |

Table 2.1: Basic boolean properties

since for

$$\mu \cdot \mu + \mu \cdot \mu' = \mu_0,$$

there are no triplets $(T, S, N)$ for which it can hold $S(T(a,a), T(a, N(a))) = 0$ for all $a$ in $[0, 1]$.

In the same vein, there are some laws that have solutions when different t-norms, t-conorms and strong negations are considered. For example, $(\mu + \mu) \cdot (\mu \cdot \mu') = \mu_0$, that comes from $(A \cup A) \cap (A \cap A^c) = \emptyset$, translated in the form $T_1(S(a,a), T_2(a, N(a))) = 0$, has infinite solutions like, for example, with an strong negation $N$, such that $N \leq N_0$, $T_1 = \min$, $T_2 = W$ and any t-conorm $S$, since $\min(S(a,a), T_2(a, N(a))) = T_2(a, N(a)) = W(a, N(a)) = \max(0, a + N(a) - 1) = 0$, because of $T_2(a, N(a)) \leq a \leq S(a,a)$, and $N(a) \leq 1 - a$, or $a + N(a) - 1 \leq 0$. 
Another case is given by the classical (derived) laws
\[ A \cap (A^c \cup B) = A \cap B, \quad A \cup (A^c \cap B) = A \cup B, \]
and the corresponding ‘possible’ fuzzy laws
\[ \mu \cdot (\mu' + \sigma) = \mu \cdot \sigma, \quad \mu + (\mu \cdot \sigma) = \mu + \sigma, \]
which functional equations
\[ T_1(a, S(N(a), b)) = T_2(a, b), \quad S_1(a, T(N(a), b)) = S_2(a, b), \]
do not have solutions with \( T_1 = T_2 \) and \( S_1 = S_2 \), respectively, but that with \( N = N_0 \) do verify
\begin{itemize}
  \item \( W(a, W^*(1 - a, b)) = \max(0, \min(a, b)) = \min(a, b) \)
  \item \( W^*(a, W(1 - a, b)) = \min(1, \max(a, b)) = \max(a, b) \)
\end{itemize}
that is, they have the solutions \( (T_1 = W, S = W^*, T_2 = \min) \) and \( (S_1 = W^*, T = W, S_2 = \max) \), respectively. Thus, it is possible to consider more-complex algebras of fuzzy sets by means of n-plas of the type \( (T_1, ..., T_m; S_1, ..., S_r; N_1, ..., N_p) \).

Notwithstanding, there are also derived laws that have no solutions neither in standard algebras, nor with different t-norms, t-conorms, or different strong negations. The fact that no standard algebra of fuzzy sets is a boolean algebra, makes impossible to simultaneously deal in such algebras with all formulas that are valid with classical sets.

### 2.3 Examples

**Example 2.3.1.** In a scale between 10 and 50 centigrade degrees, the label ‘cold’ referred to temperature, is graduated by
\[
\mu_{\text{C}}(x) = \begin{cases} 
  1, & \text{if } 10 \leq x \leq 15 \\
  \frac{25-x}{10}, & \text{if } 15 \leq x \leq 25 \\
  0, & \text{if } 25 \leq x \leq 50.
\end{cases}
\]
2.3. EXAMPLES

Which one of the following linguistic labels: cold, hot, warm, more or less cold, more or less warm, is the more adequate for the temperatures of 20, 21 and 22 centigrade degrees?

Solution. With \( C = \text{cold} \), it is \( \mu_C(20) = \frac{5}{10} = 0.5 \) and \( \mu_{\text{more or less } C}(20) = \sqrt{0.5} = 0.71 \). To obtain \( H = \text{hot} \), we can compute \( \mu_H \) as the opposite of \( \mu_C \):

\[
\mu_H(x) = \mu_C(50 + 10 - x) = \mu_C(60 - x) = \begin{cases} 
1, & \text{if } 45 \leq x \leq 50 \\
\frac{x-35}{10}, & \text{if } 35 \leq x \leq 45 \\
0, & \text{if } 10 \leq x \leq 35
\end{cases}
\]

graphically

By defining, as it is usual, warm = not cold and not hot, that is

\[
\mu_w = \mu'_\text{cold} \cdot \mu'_\text{hot}
\]

with \( \cdot \) represented by min, and \( \cdot' \) by \( N_0 \),

\[
\mu_w(x) = \min(1 - \mu_{\text{cold}}(x), 1 - \mu_{\text{hot}}(x)),
\]

for all \( x \in [10, 50] \), it results

Then:
• $x = 20$, gives $\mu_c(20) = \frac{5}{10} = 0.5$, $\mu_H(20) = 0$, $\mu_W(20) = \frac{5}{10} = 0.5$, 

• $x = 21$, gives $\mu_c(21) = \frac{4}{10} = 0.4$, $\mu_H(21) = 0$, $\mu_W(21) = \frac{6}{10} = 0.6$, 

• $x = 22$, gives $\mu_c(22) = \frac{3}{10} = 0.3$, $\mu_H(22) = 0$, $\mu_W(22) = \frac{7}{10} = 0.7$, 

and 

• $\mu_{more\ or\ less\ cold}(20) = \sqrt{\mu_c(20)} = 0.71$, 

$\mu_{more\ or\ less\ warm}(20) = \sqrt{\mu_W(20)} = 0.71$ 

• $\mu_{more\ or\ less\ ;cold}(21) = 0.63$, $\mu_{more\ or\ less\ ;warm}(21) = 0.77$, 

• $\mu_{more\ or\ less\ ;cold}(22) = 0.55$, $\mu_{more\ or\ less\ ;warm}(22) = 0.84$ 

Hence 

• The more adequate linguistic label for $x = 20$, cannot be decided but it could be either ‘not cold’, or ‘not warm’. Since, it is not hot at all, we can take ‘not cold’. 

• For $x = 21$, is ‘mol warm’ (mol= more or less) 

• For $x = 22$, is ‘mol warm’ 

Example 2.3.2. On the age of a person $p$, it is known that 

$$37 \leq Age(p) \leq 41,$$

and neither $Age(p) \leq 32$, nor $43 \leq Age(p)$. What can be said on the degree up to which it could be $Age(p) = 35$, and $Age(p) = 42$? 

Solution. What is unknown is the variation of $Age(p)$ between 32 and 37, as well as between 41 and 43. Since Age varies continuously, we can suppose there are two functions 

$$f : [32, 37] \to [0, 1], g : [41, 43] \to [0, 1]$$
such that $f(32) = 0, f(37) = 1, g(41) = 1, g(43) = 0$, with $f$ strictly non-decreasing, and $g$ strictly decreasing. Then, we can define $\mu_{Age(p)} : [0, 100] \rightarrow [0, 1]$, by

$$
\mu_{Age(p)}(x) = \begin{cases} 
0, & \text{if } x \in [0, 32] \cup [43, 100] \\
1, & \text{if } x \in [37, 41] \\
f(x), & \text{if } x \in [32, 37] \\
g(x), & \text{if } x \in [41, 43] 
\end{cases}
$$

and $\mu_{Age(p)}(35) = f(35), \mu_{Age(p)}(42) = g(42)$. Graphically

![Graph](image)

To determine $f$ and $g$ it is needed more information, but in the absence of it, we can decide to take the linear models $f(x) = \frac{x - 32}{6}$, and $g(x) = \frac{43 - x}{2}$, with which

$$
\mu_{Age(p)}(35) = \frac{3}{5}, \mu_{Age(p)}(42) = \frac{1}{2}.
$$

As it will be seen later on, 0.6 is the possibility that $Age(p) = 35$, and 0.5 that of $Age(p) = 42$. Hence, it seems a little bit more possible that it be ‘$Age(p) = 42$’ than ‘$Age(p) = 35$’.

**Example 2.3.3.** Knowing that Height(John)=175cm., and Height(Peter)=180cm, consider the two statements:

- $p = \text{It is false that John is not very tall or is more or less short}$
- $q = \text{It is false that Peter is not very tall or is more or less short}$

which is more true?

**Solution.** Both statements can be written by

- Is false that $x$ is $P$,
- with $P = \text{(not very tall) or (more or less short)}$. 

Hence

$$\mu_P(x) = S(\mu_{\text{very tall}}(x), \mu_{\text{not short}}(x)) = S(N(\mu_{\text{tall}}(x)^2), \sqrt{\mu_{\text{tall}}(A(x))})$$

with a continuous t-conorm $S$, a strong negation $N$, and a symmetry $A$ on $X$, provided $x$ varies in a scale of heights.

What should be compared are the two values $N(\mu_P(175))$ and $N(\mu_P(180))$, and for that it is needed to know $\mu_{\text{tall}}$. Let us take

$$\mu_{\text{tall}}(x) = \begin{cases} 0, & \text{if } x \in [0, 150] \\ \text{strictly non decreasing}, & \text{if } x \in [150, 190] \\ 1, & \text{if } x \in [190, 210] \end{cases}$$

with, perhaps, $\mu_{\text{tall}}(x) = 0.025x - 3.75, x \in [150, 190]$, if we need to have numbers.

Hence, with $A(x) = 210 - x$, it is $A(175) = 210 - 175 = 35$, and $\mu_{\text{tall}}(35) = 0$, as well as $A(180) = 220 - 180 = 30$, and $\mu_{\text{tall}}(30) = 0$, because of that

$$\mu_P(175) = S(N(\mu_{\text{tall}}(175)), 0) = N(\mu_{\text{tall}}(175)^2)$$
$$\mu_P(180) = S(N(\mu_{\text{tall}}(180)), 0) = N(\mu_{\text{tall}}(180)^2)$$

Since $\mu_{\text{tall}}$ is strictly non-decreasing between 150 and 190, it is $\mu_{\text{tall}}(175) < \mu_{\text{tall}}(180)$, and $N(\mu_{\text{tall}}(180)^2) < N(\mu_{\text{tall}}(175)^2)$. Finally,

$$N(N(\mu_{\text{tall}}(175)^2)) < N(N(\mu_{\text{tall}}(180)^2)), \text{ or } \mu_{\text{tall}}(175)^2 < \mu_{\text{tall}}(180)^2,$$

and $q$ is strictly more true than $p$.

Notice that it is not needed to fix $S$ and $N$, but only a form for $\mu_{\text{tall}}$, as well as to accept that $\mu_{\text{very}} P(x) = \mu_P(x)^2, \mu_{\text{not}} P(x) = \sqrt{\mu_P(x)}$, and $A(x) = 210 - x$. This last hypothesis is perfectly reasonable since $\mu_{\text{tall}}$ is non-decreasing, and then the order $\leq_{\mu_{\text{tall}}}$ is just the order of $[0, 210]$.

Provided we need to know up to which numerical degree $p$ and $q$ do hold, we can use the linear function in the figure,
and fix $N = N_0$. It results

degree up to which $p$ is true $= 1 - (1 - \mu_{\text{tall}}^2(175)) = \mu_{\text{tall}}^2(175) = 0.391$

degree up to which $q$ is true $= \mu_{\text{tall}}^2(180) = 0.563$,

that shows $q$ is much more true than $p$.

**Example 2.3.4.** It is known that the algebra with which fuzzy sets must be combined to satisfy the laws

$$\mu + \mu \cdot \sigma = \mu, \mu \cdot (\mu + \sigma) = \mu,$$

as well as that the negation is linear, Determine the triplet $(T, S, N)$ and , with $X = [0, 10]$, and

$$\mu(x) = \frac{x}{10}, \sigma(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 5 \\
\frac{x-5}{6}, & \text{if } 5 \leq x \leq 8 \\
\frac{x-6}{4}, & \text{if } 8 \leq x \leq 10,
\end{cases}$$

compute $\mu \cdot \sigma, \mu + \sigma$, and $\mu' + \sigma$.

**Solution.** The first law of absorption $\mu + \mu \cdot \sigma = \mu$, implies $S = \max$ for any $T$. The second law of absorption $\mu \cdot (\mu + \sigma) = \mu$, implies $T = \min$ for any $S$. Hence $(T, S) = (\min, \max)$, and the only linear $N$ is $N = N_0$. Hence, $(T, S, N) = (\min, \max, 1 - id)$. With the graphics of $\mu$ and $\sigma$ in the figure,
it follows \( \mu \cdot \sigma = \min(\mu, \sigma) = \sigma, \mu + \sigma = \max(\mu, \sigma) = \mu \), and \( \mu' + \sigma = (1 - \mu) + \sigma \) is the pointed curve. That is,

\[
(\mu' + \sigma)(x) = \begin{cases} 
\mu'(x), & \text{if } x \in [0, 6.87] \\
\sigma(x), & \text{if } x \in [6.87, 10].
\end{cases}
\]

Notice that \( x = 6.87 \) comes from the equation \( 1 - \frac{x}{10} = \frac{x-5}{6} \).

**Example 2.3.5.** Consider the Sugeno’s family of strong negations \( N_\lambda(x) = \frac{1-x}{1+\lambda x}(\lambda > -1) \). If \(-1 < \lambda_1 < \lambda_2\), it follows \( 1 + \lambda_1 x < 1 + \lambda_2 x \), or \( \frac{1}{1+\lambda_2 x} < \frac{1}{1+\lambda_1 x} \), that is, \( N_{\lambda_2}(x) < N_{\lambda_1}(x) \). Hence,

- If \( \lambda \leq 0 \) : \( N_0 \leq N_\lambda \)
- If \( 0 \leq \lambda \) : \( N_\lambda \leq N_0 \)

Compare the graphics of \( \mu' = N_0 \circ \mu \) and \( \mu' = N_1 \circ \mu \), in a figure, if

\[
\mu(x) = \begin{cases} 
0, & \text{if } x \in [0, 3] \cup [7, 10] \\
1, & \text{if } x \in [4, 6] \\
x - 3, & \text{if } x \in [3, 4] \\
7 - x, & \text{if } x \in [6, 7]
\end{cases}
\]
Solution. It is
\[
\mu'(x) = \begin{cases} 
1, & \text{if } x \in [0, 3] \cup [0, 4] \\
0, & \text{if } x \in [4, 6] \\
\frac{4-x}{x}, & \text{if } x \in [3, 4] \\
\frac{x-6}{x}, & \text{if } x \in [6, 7],
\end{cases}
\]
and \( \mu'_0(x) = N_1(\mu(x)) = \frac{1-\mu(x)}{1+\mu(x)} \), or
\[
\mu'_0(x) = \begin{cases} 
0, & \text{if } x \in [0, 3] \cup [7, 10] \\
1, & \text{if } x \in [4, 5] \\
\frac{4-x}{x-2}, & \text{if } x \in [3, 4] \\
\frac{x-6}{x-2}, & \text{if } x \in [6, 7]
\end{cases}
\]
Hence,

Look that \( N_1(3.2) = 0.6, N_1(3.5) = 0.3, N_1(3.4) = 0.43, N_1(3.6) = 0.25, N_1(3.8) = 0.1 \), but \( N_0(3.2) = 0.8, N_0(3.5) = 0.5, N_0(3.6) = 0.4, N_0(3.4) = 0.6, , N_0(3.6) = 0.4, N_0(3.8) = 0.2 \).

Example 2.3.6. The negation is linear, and the standard algebra must verify the law \( \mu = \mu \cdot \sigma + \mu \cdot \sigma' \). Determine the triplet \((T, S, N)\), and with \( \mu(x) = \frac{x}{10} \) (in \( X = [0, 10] \)) and

\[
\sigma(x) = \begin{cases} 
1, & \text{if } x \in [0, 5] \\
\frac{\frac{5-x}{2}}{2}, & \text{if } x \in [5, 7] \\
0, & \text{if } x \in [7, 10]
\end{cases}
\]
check that \( \mu \cdot \sigma + \mu \cdot \sigma' = \mu, \mu \cdot \sigma + \mu' \cdot \sigma = \sigma, \) and \((\mu \cdot \sigma')' = \sigma + \mu' \cdot \sigma' \).
CHAPTER 2. ALGEBRAS OF FUZZY SETS

Solution. From $N$ linear, $N = N_0$, it follows that we can take $\varphi = \text{id}$, and from $\mu \cdot \sigma + \mu \cdot \sigma' = \mu$ it follows $T = \text{prod } \varphi$, $S = W^\varphi$, and $N = N_\varphi$. Hence $(T, S, N) = (\text{prod}, W^\varphi, N)$.

Since $\mu'(x) = 1 - \frac{x}{10}$, and $\sigma'(x) = \left\{ \begin{array}{ll}
\frac{x}{10} & 
\frac{5}{2} \\
0
\end{array} \right.$, it follows

$$
\mu \sigma(x) = \left\{ \begin{array}{ll}
\frac{x}{10} & 
\frac{5}{2} \\
0
\end{array} \right., \quad \mu \sigma'(x) = \left\{ \begin{array}{ll}
0 & 
\frac{x}{20} \\
\frac{x}{10}
\end{array} \right., \mu' \sigma(x) = \left\{ \begin{array}{ll}
1 - \frac{x}{10} & 
\frac{7-x}{2} (1 - \frac{x}{10}) \\
0
\end{array} \right.
$$

Hence,

$$(\mu \cdot \sigma + \mu \cdot \sigma')(x) = \left\{ \begin{array}{ll}
W^\varphi(\frac{x}{10}, 0) = \frac{x}{10} & 
W^\varphi(\frac{x(7-x)}{20}, \frac{x(x-5)}{20}) = \frac{x}{10} \\
W^\varphi(0, \frac{x}{10}) = \frac{x}{10}
\end{array} \right. = \frac{x}{10} = \mu(x),$$

$$(\sigma \cdot \mu + \sigma \cdot \mu')(x) = \left\{ \begin{array}{ll}
W^\varphi(\frac{x}{10}, 1 - \frac{x}{10}) = 1 & 
W^\varphi(\frac{x(7-x)}{20}, \frac{7-x}{2} (1 - \frac{x}{10}) = \frac{7-x}{2} \\
W^\varphi(0, 0) = 0
\end{array} \right. = \sigma(x).$$

Finally, since,

$$(\mu \sigma')(x) = 1 - (\mu \cdot \sigma')(x) = \left\{ \begin{array}{ll}
1 & 
1 - \frac{x(x-5)}{20} \\
1 - \frac{x}{10}
\end{array} \right., \text{ and } (\mu' \cdot \sigma')(x) = \left\{ \begin{array}{ll}
0 & 
(1 - \frac{x}{10}) \frac{x-5}{2} \\
1 - \frac{x}{10}
\end{array} \right.,$$

it results

$$(\sigma + (\mu' \cdot \sigma'))(x) = \left\{ \begin{array}{ll}
W^\varphi(1, 0) = 1 & 
W^\varphi(\frac{7-x}{2}, \frac{x-5}{2} (1 - \frac{x}{10}) = 1 - \frac{x(x-5)}{20} = (\mu \cdot \sigma')(x) \\
W^\varphi(0, 1 - \frac{x}{10}) = 1 - \frac{x}{10}
\end{array} \right. = (\sigma + (\mu' \cdot \sigma'))(x).$$

Example 2.3.7. Predicate $F = \text{ high fever}$ refers to the interval $[37, 42]$ in a clinical thermometer, in which the values $\{37, 37.5, 38, \ldots 41.5, 42\}$ are significative. Asking an expert one obtains the following fuzzy set

$$
\mu_F = 0.3/38.5 + 0.5/39 + 0.7/39.5 + 0.8/40 + 0.9/40.5 + 1/41 + 1/41.5 + 1/42
$$
where it is clear that $0/37 + 0/37.5 + 0/38$ is avoided since this values of the body's temperature are not significative for $F$. With all that, give the membership function of $P$=very high fever, $Q$=more or less high fever, $R$=low fever, $S$=not high fever.

Solution. With the usual definition $\mu_{\text{very } F} = \mu_F^2$, $\mu_{\text{not } F} = +\sqrt{\mu_F}$, $\mu_{\text{low } F} = \mu_F(37 + 42 - x) = \mu_F(79 - x)$, $\mu_{\text{not } F} = 1 - \mu_F$, it results:

- $\mu_P = 0.09/38.5 + 0.25/39 + 0.49/39.5 + 0.64/40 + 0.81/40.5 + 1/41 + 1/41.5 + 1/42$.
- $\mu_Q = 0.55/38.5 + 0.7/39 + 0.84/39.5 + 0.89/40 + 0.95/40.5 + 1/41 + 1/41.5 + 1/42$.
- $\mu_R = 1/37 + 1/37.5 + 1/38 + 0.9/38.5 + 0.8/39.5 + 0.7/39.5 + 0.5/40.5 + 0.3/40.5$.
- $\mu_S = 1/37 + 1/37.5 + 1/38 + 0.7/38.5 + 0.5/39.5 + 0.3/39.5 + 0.2/40.5 + 0.1/40.5$.

Notice the incoherence produced by $\mu_S = \mu_{\text{not } F} \leq \mu_{\text{low } F} = \mu_R$. An incoherence showing that it cannot be taken the representation $\mu_{\text{not } F} = 1 - \mu_F$, but some $\mu_{\text{not } F} = N \circ \mu_F$ with $N \geq N_0$.

For example, if $N(x) = \frac{1-x}{1-0.9x}$, it is

$\mu_S = 1/37 + 1/37.5 + 1/38 + 0.96/38.5 + 0.91/39.5 + 0.81/39.5 + 0.71/40 + 0.53/40.5 + 1/41 + 1/41.5 + 1/42$, showing $\mu_{\text{low } F} \leq \mu_{\text{not } F}$.

Look that

$\mu_{F \& \text{low } F} = 0.3/38.5 + 0.5/39.5 + 0.7/39.5 + 0.5/40 + 0.3/40.5$

provided $\mu_{F \& \text{low } F} = \min(\mu_F, \mu_{\text{low } F})$.

**Example 2.3.8.** Describe in fuzzy terms, the statements
\[ p= \text{John is young and around forty}, \]
\[ q= \text{John is old or around forty} \]

*Solution.* The solution will come after representing the predicates \( P=\text{young}, \)
\( aP=\text{old}, \) \( A40=\text{around forty}, \) in a scale of 0 to 100 years. The general forms of \( \mu_P, \mu_{aP}, \) and \( \mu_{A40}, \) are

\[
\begin{align*}
\mu_P(x) & = \mu_P(100 - x) \quad \text{since } \mu_P \text{ is non-decreasing.} \\
\text{Degree}(p) & = T(\mu_P(x), \mu_{A40}(x)), \quad \text{Degree}(q) = S(\mu_{aP}(x), \mu_{A40}(x)),
\end{align*}
\]

with convenient continuous t-norm \( T \) and t-conorm \( S. \) This formulas are the description of \( p \) and \( q \) in fuzzy terms.

For example, if \( a = 20, b = 50, \mu_P = \frac{50-x}{30}, \) if \( 20 \leq x \leq 50, \) and \( 40 - a = 30, 40 + b = 50, \) with \( \mu_{A40} \) piece-wise linear, \( T = \min, \) \( S = \max, \) with

\[ 
\mu_{aP}(x) = \mu_P(100 - x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq 50 \\
\frac{x-50}{30}, & \text{if } 50 \leq x \leq 80 \\
1, & \text{if } 80 \leq x \leq 100,
\end{cases}
\]

the graphics is
The slashed function describes $p$, and the continuous one describes $q$. Of course, $\text{Degree}(p) \leq \min(\mu_P(x), \mu_{A40}(x)) \leq \mu_P(35) = 0.5$.

Degree($p$) $\leq \min(\mu_P(x), \mu_{A40}(x)) \leq \mu_P(35) = 0.5$.

### 2.3.1

Degree($p_1$) = $\mu_P(45) = \frac{5}{30} = \frac{1}{6}$, Degree($p_2$) = $\mu_{aP}(45) = \mu_P(55) = \frac{5}{30} = \frac{1}{6}$, Degree($p_3$) = $\mu_{A40}(45) = \frac{50-45}{10} = \frac{1}{2}$.

To obtain the Degree($p_4$), we need to complete $\mu_{\text{not young and not very old}}(x) = \min(\mu_P(x), \mu_{aP}(x)^2)$, that gives Degree($p_4$) = $\min(\mu_P(45), 1-\mu_P(100-45)^2) = 1-\max(\frac{1}{6}, 1-\frac{1}{6}) = 1-\max(\frac{1}{6}, \frac{35}{36}) = \frac{1}{36}$. Analogously, for having Degree($p_5$), it is needed to complete $\mu_{\text{very young or not old}}(x) = \max(\mu_P(x)^2, \sqrt{\mu_{aP}(x)}) = \max(\frac{1}{36}, \sqrt{\frac{1}{6}}) = \frac{\sqrt{6}}{6}$. Finally, for $p_6$, it is $\mu_{\text{not middle aged}}(45) = 1-\mu_{\text{middle aged}}(45) = 1-\min(1-\mu_P(45), 1-\mu_{aP}(45)) = \max(\mu_P(45), \mu_{aP}(45)) = \max(\frac{1}{6}, \frac{1}{6}) = \frac{1}{6}$, that also gives

$$\mu_{\text{middle aged}}(45) = 1-\frac{1}{6} = \frac{5}{6}.$$  

Since $\frac{5}{6} > \frac{1}{2} > \frac{\sqrt{6}}{6} > \frac{1}{6} > \frac{1}{36}$, it results that “around forty” is the more appropriate of the five labels, but since $\frac{5}{6}$ is much more greater again, the more appropriate is out of the given list, it is ‘middle aged’.

**Remark 2.3.9.** To select $T$ and $S$, the following points could be taken into account,
• It could be perfectly the case that 'John is young and not young' with a positive degree. Hence, the laws of Non-contradiction can be avoided, and $T \notin \{W\}$.

• Since it is reasonable to accept that 'John is old' and 'John is old' does coincide with 'John is old', and 'John is young' or 'John is young' does coincide with 'John is young', we can decide to take either $T = \min, S = \max$, or $T$ and $S$ as ordinal-sums.

• Provided idempotency is avoided, e.g., 'John is young' and 'John is young' does coincide with 'John is very young', instead of $T = \min$, we can take $T = \text{prod}$, that is more interactively than min. In this case, because it does not seems that duality should be avoided, we could take $S = \text{Prod}^*$, and then

$$\mu_{\text{very young or not old}}(45) = \text{Prod}^*(\frac{1}{3}, \frac{\sqrt{6}}{6}) = \frac{1}{6}(1 + \frac{35\sqrt{6}}{36}) = 0.5636,$$

that is greater than the value $\frac{\sqrt{6}}{6} = 0.408$ obtained with $= \max$.

\[2.3.2\]

In $X = [0, 10]$ the predicate $P = \text{big}$ is represented by $\mu(x) = \frac{x}{10}$. In which points in $[0, 10]$ is the degree of 'big' is less than that of 'not big'? 

Solution. Given $\mu$, the problem is to find for which $x \in X$ it is $\mu(x) \leq \mu'(x) = N_\varphi(\mu(x)) = \varphi^{-1}(1 - \varphi(\mu(x)))$, that is, $\mu(x) \leq \varphi^{-1}(\frac{1}{2})$. Then,

• If $N = N_0$, $\frac{x}{10} \leq 1 - \frac{x}{10}$, or $x \leq 5$.

• If $N = N_1$, $\frac{x}{10} \leq \frac{1 - \frac{x}{10}}{1 + \frac{x}{10}}$, or $x^2 + 20x - 100 \leq 0$, that means $x \leq 10(\sqrt{2} - 1) = 4.142$

• If $N = N_2$, $\frac{x}{10} \leq \frac{1 - \frac{x}{10}}{1 + \frac{x}{10}}$, or $x^2 + 10x - 50 \leq 0$, that means $x \leq \sqrt{75} - 5 = 3.66$

Hence,
2.4. ON AGGREGATING IMPRECISE INFORMATION

- If $N = N_0$, the set is $[0, 5]$, and the threshold (of selfcontradiction) of big is 5.
- If $N = N_1$, the set is $[0, 4.142]$, and the threshold is 4.142
- If $N = N_2$, the set is $[0, 3.66]$, and the threshold is 3.66

Notice that changing big by not big, the thresholds do remain but the sets are, respectively, $[5, 10]$, $[4.142, 10]$, and $[3.66, 10]$

2.4 On aggregating imprecise information

2.4.1

The kind of problems this section will deal with is the following. An exam is corrected by three referees $R_1, R_2, R_3$, each one with a different weight of strongness $W(R_i) \in [0, 1]$, $1 \leq i \leq 3$, such that $\sum_{i=1}^{3} W(R_i) = 1$. Each referee assigns a numerical qualification $p_i \in [0, 10]$ to the exam delivered by a given student. How these qualification can be “aggregated” to obtain final qualification for the student’s exam? A recognized usual way of doing it is by the weighted mean:

$$\frac{1}{10} Q = \frac{p_1}{10} W(R_1) + \frac{p_2}{10} W(R_2) + \frac{p_3}{10} W(R_3),$$

with $\frac{p_i}{10} \in [0, 10]$. For example, if $W = (0.5, 0.3, 0.2)$ and $P = (7, 6, 5)$, it follows

$$\frac{1}{10} Q = 0.7 \times 0.5 + 0.6 \times 0.3 + 0.2 \times 0.5 = 0.63$$

that implies $Q = 6.3$. Provided the three referees have the same weight, it is $W = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and then, $Q = p_1 \cdot \frac{1}{3} + p_2 \cdot \frac{1}{3} + p_3 \cdot \frac{1}{3} = \frac{p_1 + p_2 + p_3}{3} = \frac{7 + 6 + 5}{3} = 6$, is just the arithmetic mean of the three qualifications.

Another way of obtaining the final qualification, this time by ignoring the referee’s character, is by the geometric mean

$$Q = \sqrt[3]{p_1 \cdot p_2 \cdot p_3} = \sqrt[3]{7 \times 6 \times 5} = 5.94,$$
CHAPTER 2. ALGEBRAS OF FUZZY SETS

showing that in a problem with \( p_1 = p_2 = 10, p_3 = 0 \), it results \( Q = \sqrt[3]{10 \times 10 \times 0} = 0 \), when the arithmetic mean is \( \frac{20}{3} = 6.67 \).

2.4.2

Most of these problems are “represented” by the so-called Aggregation Functions, that is, functions

\[ A : [0, 1]^n \to [0, 1], \]

such that

1. \( A \) is continuous in all variables
2. \( A(0, \ldots, 0) = 0 \), and \( A(1, \ldots, 1) = 1 \)
3. If \( x_1 \leq y_1, \ldots, x_n \leq y_n \), then \( A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n) \)

Sometimes it is said that \( A \) is an \( n \)-dimensional aggregation function. Continuous t-norms and continuous t-conorms are 2-dimensional aggregation functions.

Of the many many types of aggregation function, a particular and important type are the quasi-linear means,

\[ M(x_1, \ldots, x_n) = f^{-1} \left( \sum_{i=1}^{n} p_i f(x_i) \right) \]

with \((p_1, \ldots, p_n)\) in \([0, 1]\), verifying \( \sum_{i=1}^{n} p_i = 1 \), and \( f : [0, 1] \to \mathbb{R} \), continuous, one-to-one, and monotonic. Function \( f \) is called the generator of \( M \).

Notice that if \( f \) is the identity \( f(x) = x \), we get the weighted means:

\[ M(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i, \]

that with \( p_1 = \frac{1}{n} \) \((1 \leq i \leq n)\) is the arithmetic mean

\[ M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i, \]
and with \( f(x) = -\log x \), and \( p_i = \frac{1}{n} \) \( (1 \leq i \leq n) \) is the geometric mean

\[
M(x_1, \ldots, x_n) = \sqrt[n]{p_1 \cdot p_2 \cdots p_n}.
\]

With \( f(x) = x^\alpha \) \( (\alpha > 0) \), is \( f^{-1}(x) = x^{\frac{1}{\alpha}} \), and with \( p_1 = \frac{1}{n} \), it is obtained the family of quasi-linear means,

\[
M_\alpha(x_1, \ldots, x_n) = \left( \frac{x_1^\alpha + \ldots + x_n^\alpha}{n} \right)^{\frac{1}{\alpha}}.
\]

In particular, with \( \alpha = 1 \), is \( M_1 \) the arithmetic mean, and with \( \alpha = -1 \), it follows

\[
M_{-1}(x_1, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}} \quad (\text{if } x_1, \ldots, x_n \neq 0),
\]

called Harmonic Mean. As it is easy to prove,

\[
\begin{align*}
\lim_{\alpha \to 0} M_\alpha(x_1, \ldots, x_n) &= \sqrt[n]{x_1 \cdots x_n} \\
\lim_{\alpha \to \infty} M_\alpha(x_1, \ldots, x_n) &= \max(x_1, \ldots, x_n) \\
\lim_{\alpha \to -\infty} M_\alpha(x_1, \ldots, x_n) &= \min(x_1, \ldots, x_n)
\end{align*}
\]

2.4.3

It is said that \( M : [0,1]^n \to [0,1] \) is a mean, when \( M \) is continuous, monotonic, and verifies:

\[
\min(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n).
\]

Since, \( \min(0, \ldots, 0) \leq M(0, \ldots, 0) \leq \max(0, \ldots, 0) = 0 \), it results \( M(0, \ldots, 0) = 0 \). Since, \( \min(1, \ldots, 1) \leq M(1, \ldots, 1) \leq \max(1, \ldots, 1) = 1 \), it results \( M(1, \ldots, 1) = 1 \). Hence, of course, quasi-linear means are means, but there are such more means. An important and useful example are the Ordered Weighted Means (OWA). Its definition is the following:

\[
O : [0,1]^n \to [0,1] \text{ is an OWA, if } O(x_1, \ldots, x_n) \text{ is obtained under the process,}
\]
\textbullet{} Select weights $p_1, \ldots, p_n$ in $[0, 1]$, such that $\sum_{i=1}^{n} p_i = 1$.

\textbullet{} Permute the n-pla $(x_1, \ldots, x_n)$, to the n-pla $(x_1, \ldots, x_n)$ such that $x_1 \leq \ldots \leq x_n$.

\textbullet{} $O(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i$.

For example, if $n = 2$,

$$O(x_1, x_2) = p_1 \cdot \min(x_1, x_2) + p_2 \cdot \max(x_1, x_2), \text{ with } p_1 + p_2 = 1.$$ 

If $n = 4$, and the weights are $(0.2, 0.4, 0.3, 0.1)$, it is

$$O(0.2, 0.5, 0.7, 0.3) = O(0.2, 0.3, 0.5, 0.7) = 0.2 \times 0.2 + 0.4 \times 0.3 + 0.3 \times 0.5 + 0.1 \times 0.7 = 0.74.$$ 

2.4.4

Because they are associative, continuous t-norms and continuous t-conorms can be extended to n-dimensional aggregation functions. For example, with $n = 3$,

$$T(x_1, x_2, x_3) = T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3) = \ldots$$

$$S(x_1, x_2, x_3) = S(x_1, S(x_2, x_3)) = S(S(x_1, x_2), x_3) = \ldots$$

Nevertheless, not all aggregation functions are associative. For example, if $M$ is the arithmetic mean, $M(x_1, M(x_2, x_3)) = \frac{2x_1 + x_2 + x_3}{4}$, but $M(M(x_1, x_2), x_3) = \frac{x_1 + x_2 + 2x_3}{4}$. Concerning means, the only associative are min, and max.

In general, Aggregation Functions are not commutative. For example, a 2-dimensional quasi-linear mean

$$M(x_1, x_2) = f^{-1}(p_1 f(x_1) + p_2 f(x_2)), \text{ } p_1 + p_2 = 1,$$

is commutative if and only if $p_1 = p_2 = \frac{1}{2}$. Arithmetic and geometric means are commutative, but weighted means in general are not.
2.4. ON AGGREGATING IMPRECISE INFORMATION

2.4.5

If $T$ is a continuous t-norm, and $S$ a continuous t-conorm, the function

$$A(x_1, x_2) = p_1 T(x_1, x_2) + p_2 S(x_1, x_2), \quad p_1 + p_2 = 1$$

is an aggregation function that, since $T \leq \min \leq \max \leq S$, in general is not a mean. The only exception is with $T = \min$, and $S = \max$, as it was said before. For example,

- $A(x_1, x_2) = 0.7x_1, x_2 + 0.3W^*(x_1, x_2)$
- $A(x_1, x_2) = 0.6 \min(x_1, x_2) + 0.4(x_1 + x_2 - x_1, x_2)$
- $A(x_1, x_2) = 0.6W(x_1, x_2) + 0.4 \max(x_1, x_2)$,

are aggregation functions.

2.4.6

The pointwise aggregation of classical sets is not, in general, a classical set, but a fuzzy one. For example, the arithmetic mean verifies

$$M(0, 0) = 0, M(0, 1) = M(1, 0) = \frac{1}{2}, M(1, 1) = 1$$

and, if $A, B$ are crisp subsets, $M(A, B)$ is not a crisp subset if given by $M(\mu_A, \mu_B)(x) = M(\mu_A(x), \mu_B(x))$. On the contrary, with the geometric mean $G$, it is

$$G(0, 0) = G(0, 1) = G(1, 0) = 0, G(1, 1) = 1,$$

and $G(A, B)$ is a crisp set.

In all cases, if $\mu \in [0, 1]^X, \sigma \in [0, 1]^Y$, and $A$ is an aggregation function, then

$$A(\mu, \sigma)(x, y) = A(\mu(x), \sigma(y)),$$
for all $x \in X, y \in Y$, is a fuzzy set $A(\mu, \sigma) \in [0, 1]^{X \times Y}$ called the aggregation of $\mu$ and $\sigma$. When $X = Y$ it could be defined the fuzzy set $A(\mu, \sigma) \in [0, 1]^X$,

$$A(\mu, \sigma)(x) = A(\mu(x), \sigma(x)),$$

for all $x \in X$.

**Example 2.4.1.** If $X = \{1, 2, 3, 4, 5\}$, and $\mu = 0.6/1 + 0.7/2 + 0.5/3 + 1/4, \sigma = 0.9/1 + 0.5/3 + 0.7/4 + 0.8/5$, compute $M(\mu, \sigma), G(\mu, \sigma), \text{and } O(\mu, \sigma)$ with $O$ the OWA with weights $p_1 = 0.4, p_2 = 0.6$.

**Solution.**

$$M(\mu, \sigma) = 0.75/1 + 0.35/2 + 0.5/3 + 0.85/4 + 0.4/5$$

$$G(\mu, \sigma) = 0.735/1 + 0/2 + 0.5/3 + 0.837/4 + 0/5$$

$$O(\mu, \sigma) = (0.4 \times 0.6 + 0.6 \times 0.9)/1 + (0.4 \times 0 + 0.6 \times 0.7)/2 + (0.4 \times 0.5 + 0.6 \times 0.5)/3 + (0.4 \times 0.7 + 0.6 \times 1)/4 + (0.4 \times 0 + 0.6 \times 0.8)/5 = 0.72/1 + 0.42/2 + 0.5/3 + 0.88/4 + 0.48/5$$

Notice that $G(\mu, \sigma) \leq M(\mu, \sigma)$, but that neither $G(\mu, \sigma)$ and $O(\mu, \sigma)$, nor $M(\mu, \sigma)$ and $O(\mu, \sigma)$, are order-comparable.

**Example 2.4.2.** A linguistic variable has the fuzzy values $H = \text{high}, S = \text{short}$ and $M = \text{medium}$, with $H$ represented by

$$\mu_H(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 5 \\ 1, & \text{if } 7 \leq x \leq 10 \\ \frac{x-5}{2}, & \text{if } 5 \leq x \leq 7. \end{cases}$$

In the two suppositions, $M = H^*, S^*$, and that $M$ is the aggregation of $H$ and $S$ under the weighted mean $A(x_1, x_2) = 0.3x_1 + 0.7x_2$, compute $\mu_M$.

**Solution.** Since $S$ is an antonym of $H$, and $\mu_H$ is monotonic, $\mu_S(x) = \mu_H(10 - x)$, that is

$$\mu_S(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 3 \\ 0, & \text{if } 5 \leq x \leq 10 \\ \frac{5-x}{2}, & \text{if } 3 \leq x \leq 5. \end{cases}$$
2.4. ON AGGREGATING IMPRECISE INFORMATION

The solution for the first supposition appears in the following sequence of figures (with $\cdot = \min, 1 = 1 - id$)

\[ \mu_H \]

\[ \mu_S \]

The solution for the second supposition is

\[ \mu_H(x) = \mu_{A(H,S)}(x) = \begin{cases} 
0.7, & \text{if } 0 \leq x \leq 3 \\
\frac{0.7(5-x)}{2}, & \text{if } 3 \leq x \leq 5 \\
\frac{0.3(x-5)}{2}, & \text{if } 5 \leq x \leq 7 \\
0.3, & \text{if } 7 \leq x \leq 10,
\end{cases} \]

graphically,
Chapter 3

Reasoning and approximate reasoning

3.1 What does it mean “logic”?

The question is, in fact, a philosophical one whose discussion does not correspond to this text, and that received a lot of comments and discussions by philosophers. Instead of such question, there is the more particular, what is a logic?, that can be answered not in philosophical but in technical terms only requiring of just the mathematical definition of what is a consequence’s operator. A definition that corresponds to an abstraction of the term “deduction”.

3.1.1

A logic is a triplet \((X, \mathcal{A}, C)\), where \(\mathcal{A} \subseteq \mathcal{P}(X)\) is a family of parts of \(X\), and \(C : \mathcal{A} \rightarrow \mathcal{A}\) is a mapping verifying

1. For all \(P \in \mathcal{A}, P \subseteq C(P)\)
2. If \(P, Q \in \mathcal{A}\), and \(P \subseteq Q\), then \(C(P) \subseteq C(Q)\)
3. For all \(P \in \mathcal{A}, C(C(P)) \subseteq C(P)\).
It can be said that the pair \((A, C)\) defines a logic in the set \(X\).

From 1. and 2., it follows \(C(P) \subseteq C(C(P))\), and from 3. results

3’. For all \(P \in A, C(C(P)) = C(P)\), or \(C^2 = C\).

Hence, a consequence’s operator is one that is extensive (1.), monotonic (2.), and a closure (3’). These operators were introduced by the logician Alfred Tarski in the thirties of XX Century, and are called to be compact provided for all \(P \in A\), it exists a finite set \(\{p_1, \ldots, p_n\} \subseteq P\) such that

4. \(C(P) = C\{p_1, \ldots, p_n\}\).

It is clear that if \(P\) is finite, \(C\) is compact. This happens, for example if \(X\) is finite.

A consequence’s operator \(C\) is consistent when \(q \in C(P) \Rightarrow q \notin C(P)\).

**Example 3.1.1.** In a finite lattice \((X, \cdot, +; 0, 1)\), let’s consider \(A = P_0(X) = \{P \subseteq X; P = \{p_1, \ldots, p_n\}, p_\wedge = p_1 \cdots p_n \neq 0\}\), and the operator

\[\text{Cons} : P_0(X) \to P_0(X),\text{ defined by } Cons(P) = \{q \in X; p_\wedge \leq q\},\]

with the partial order \(\leq\) given by \(x \leq y \iff x \cdot y = x\). \(\text{Cons}\) is a consequence’s operator:

1. Since \(p_\wedge \leq p_i\), it is \(p_i \in Cons(p)\), for all \(p_i \in P\). That is, \(P \subseteq Cons(P)\).

2. If \(P = \{p_1, \ldots, p_n\} \subseteq \{p_1, \ldots, p_n, p_{n+1}, \ldots, p_m\} = Q\), since \(q_\wedge = p_1 \cdots p_n \cdot p_{n+1} \cdots p_m \leq p_1 \cdots p_n = p_\wedge\), if \(q \in Cons(P)\), from \(p_\wedge \leq q\), it follows \(q_\wedge \leq q\), and \(q \in Cons(Q)\) hence, \(P \subseteq Q\) implies \(Cons(P) \subseteq Cons(Q)\).

3. Obviously, Min Cons\((P) = p_\wedge\), hence, Cons\((P) \in P_0(X)\) since Min Cons\((P) \neq 0\). Then, if \(q \in Cons(Cons(P))\), or Min Cons\((P) \leq q\), or \(p_\wedge \leq q\), it follows \(q \in Cons(P)\). Hence, Cons\((Cons(P)) \subseteq Cons(P)\).
3.1. WHAT DOES IT MEAN “LOGIC”?

In any finite lattice \((X, \cdot, +; 0, 1)\), it can be considered the logic \((X, \mathcal{P}_0(X), \text{Cons})\), and it can be proved that if the lattice is endowed with a complement ' such that \((X, \cdot, +'; 0, 1)\) is a boolean algebra, any operator of consequences \(C : \mathcal{P}_0(X) \rightarrow \mathcal{P}_0(X)\) verifies \(C \subset \text{Cons}\), that is \(C(P) \subset \text{Cons}(P)\), for all \(P \in \mathcal{P}_0(X)\). \(\text{Cons}\) is the higher operator of consequences in a boolean algebra with \(\mathcal{A} = \mathcal{P}_0(X)\).

The set of premises \(P\) is consistent, if \(p_\wedge\) is not self contradictory (\(p_\wedge \not\leq p'_\wedge\)). If it were \(P_\wedge \leq P'_\wedge\), the \(P_\wedge \in \text{Cons}(P)\) and also \(P'_\wedge \in \text{Cons}(P)\), that is absurd. In this cases since \(q \in \text{Cons}(P)\) and \(q' \in \text{Cons}(P)\), or \(p_\wedge \leq q\) and \(p_\wedge \leq q'\) (or \(q \leq p'\wedge\)) implies \(p_\wedge \leq p'_\wedge\), that is absurd. This \(q \in \text{Cons}(P)\) does imply \(q' \notin \text{Cons}(P)\).

**Remark 3.1.2.** Instead of a lattice, let us take the set \([0, 1]^X\) endowed with a fuzzy intersection

\[\mu_\wedge = \mu_1 \cdots \mu_n = T \circ (\mu_1 \times \ldots \times \mu_n)\]

\((T\) a continuous t-norm), the partial order \(\mu \leq \sigma \iff \mu(x) \leq \sigma(x)\), for all \(x \in X\), and the empty set \(\mu_0 = \mu_{\emptyset}\). Take the set \(\mathcal{P}_0([0, 1]^X)\) that consists of the finite subsets \(P \subset [0, 1]^X\) such that \(\mu_\wedge \neq \mu_0\). The definition

\[\text{Cons}(P) = \{\sigma \in [0, 1]^X; \mu_\wedge \leq \sigma\},\]

if \(P = \{\mu_1, \ldots, \mu_n\}\) gives the same as in the case before.

There is a, perhaps alternative, way of constructing a logic in a set \(X\). It follows from the following results.

- If \((X, \mathcal{A}, C)\) is a logic in \(X\), the binary relation ‘\(x \leq_C y \iff y \in C(\{x\})\)’, defined if \(\{x\} \in \mathcal{A}\), is a preorder.

**Proof.** Since \(\{x\} \subset C(\{x\})\), it is \(x \leq_C y\), for all \(\{x\} \in \mathcal{A}\). If \(x \leq_C y\) and \(y \leq_C z\), it is \(y \in C(\{x\})\), and \(z \in C(\{y\})\), hence from \(\{y\} \subset C(\{x\})\) follows \(C(\{y\}) \subset C(C(\{x\})) = \{x\}\), and \(z \in C(\{x\})\), or \(x \leq_C z\). \(\Box\)

Notice that the preorder \(\leq_C\) is defined only with the pairs \((x, y) \in X \times X\) such that \(\{x\} \in \mathcal{A}\) and \(\{y\} \in \mathcal{A}\).
• Given a set $X$ and $A \subseteq \mathcal{P}(X)$, if $\leq$ is a preorder in $X$ such that

If $x \in P$, and $x \leq y$, it is $y \in Q$, for some $Q \subseteq A$,

the operator $C_{\leq} : A \to A$, defined by $C_{\leq}(P) = \{y \in X ; \exists x \in P \land x \leq y\}$, verifies,

1. $\forall x \in P$, it is $x \leq x$, then $x \in C_{\leq}(P) : P \subseteq C_{\leq}(P)$

2. If $P \subseteq Q$, and $y \in C_{\leq}(P)$, there is $x \in P$ such that $x \leq y$. Since

   $x \in Q$, it is $y \in C_{\leq}(Q)$. Hence, $C_{\leq}(P) \subseteq C_{\leq}(Q)$.

3. If $y \in C_{\leq}(C_{\leq}(P))$, there is $x \in C_{\leq}(P)$, such that $x \leq y$. But there

   is also $z \in P$ such that $z \leq x$. Hence $z \leq y$, and $y \in C_{\leq}(P)$. That

   is, $C_{\leq}(C_{\leq}(P)) = C_{\leq}(P)$.

4. The preorder $\leq_{C_{\leq}}$ does coincide with the initial one $\leq$, since $x \leq_{C_{\leq}}$

   $y \iff y \in C_{\leq}(\{x\}) \iff x \leq y$.

**Example 3.1.3.** Let it be the set $X = \{1, 2, 3, 4, 5\}$ endowed with the preordering

![Diagram](https://via.placeholder.com/150)

It is, for example $C_{\leq}(\{1, 5\}) = \{1, 5, 2, 3, 4\}, C_{\leq}(\{4, 5\}) = \{4, 5\}, C_{\leq}(\{2, 3, 4\}) = \{2, 3, 4, 5\}, C_{\leq}(\{1, 2, 4, 5\}) = \{1, 2, 4, 5, 3\} = X, C_{\leq}(\{5\}) = \{5\}$, and $C_{\leq}(\{3\}) = \{3, 4, 5\}$.

Hence, and to some extent, it is possible to identify a logic in a set with a preordering of the set.

### 3.1.2

A process to pass from a set of premises $P = \{\mu_1, \ldots, \mu_n\}$ to a ‘conclusion’ $\sigma$ is a *conclusive reasoning*, that is sometimes symbolized by $P \vdash \sigma$. A
3.1. WHAT DOES IT MEAN “LOGIC”? conclusive reasoning $P \vdash \sigma$ is deductive if there exists either an operator of consequences $C$, or a preorder $\preceq$, such that $\sigma \in C(P)$, or $\mu_i \leq \sigma$ for some $\mu_i \in P$.

Anyway, not all conclusive reasonings are deductive. In common reasoning if there is the possibility of stating $P \vdash \sigma$ and defining $C(P) = \{\sigma; P \vdash \sigma\}$, the axiom of monotonicity is not always verified by such $C$, but it could verify

- If $P \subseteq Q$, then $C(Q) \subseteq C(P)$, $C$ is anti-monotonic.

- If $P \subseteq Q$, then $C(P)$ and $C(Q)$ are not comparable, that is, it is neither $C(P) \subseteq C(Q)$, nor $C(Q) \subseteq C(P)$, $C$ is non-monotonic.

Most common conclusive reasonings are not deductive, but of a conjectural type, that is, the conclusion $\sigma$ is provisionally accepted because, simply, it is non contradictory with all or with part of the information. It should be noticed that the lost of the monotonicity of $C$ motivates the lost of the transitive property of the preorder $\preceq_C$. We will say that $P \vdash \sigma$ is a conjectural kind of conclusive reasoning if there exists an operator of consequences $C$, such that

$$P \vdash \sigma \iff \sigma' \notin C(P) \iff N \circ \sigma \notin C(P),$$

for some strong negation $N$. Analogously, if what we have is a preorder $\preceq$ (instead of $C$), $\sigma$ is a conjecture of the information contained in $P$, when

$$P \vdash \sigma \iff \forall \mu \in P : \mu \npreceq \sigma'.$$

In what follows, we will only take into account the operator $Cons(P) = \{\sigma; \mu_\wedge \leq \sigma\}$, provided $\mu_\wedge \neq \mu_0$. Consequently, the set of the conjectures that, through $Cons$, is associated to any finite set $P = \{\mu_1, \ldots, \mu_n\}$ of premises such that $\mu_\wedge = \mu_1 \cdots \mu_n = T \circ (\mu_1 \times \ldots \times \mu_n) \neq \mu_0$, for some continuous t-norm $T$, is

$$\text{Conj}(P) = \{\sigma \in [0, 1]^X; \mu_\wedge \leq \sigma'\}^c,$$
with \( \sigma' = N \circ \sigma \), for some strong negation \( N \).

Notice that it is necessary to take a connectives pair \((T, N)\) to define \( \text{Conj}(P) \).

**Remark 3.1.4.** Instead of \( \mu_\land \neq \mu_0 \), in what follows we will suppose that \( \mu_\land \) is not self-contradictory, that is \( \mu_\land \not\leq \mu'_\land \). With this hypothesis, \( \sigma \in \text{Cons}(P) \iff \mu_\land \leq \sigma \), it can’t be \( \mu_\land \leq \sigma' \) because of \( \sigma' \leq \mu'_\land \) implies \( \mu_\land \leq \mu'_\land \). Hence, it should be \( \mu_\land \not\leq \sigma' \) or \( \sigma \in \text{Conj}(P) \). That is, \( \text{Cons}(P) \subset \text{Conj}(P) \).

In addition; \( \text{Cons} \) is consistent. Notice that \( \mu_0 \notin \text{Conj}(P) \).

In fact, \( \mu_\land \not\leq \mu'_\land \) is more general than \( \mu_\land \neq \mu_0 \), since \( \mu_\land = \mu_0 \) implies \( \mu_\land = \mu_0 \leq \mu_1 = \mu'_0 = \mu'_\land \). Hence, if it is \( \mu_\land \not\leq \mu'_\land \), it is also \( \mu_\land \neq \mu_0 \).

With this change,

\[
\text{Conj}(P) = \text{Cons}(P) \cup \text{Hyp}(P) \cup \text{Sp}(P),
\]

where

- \( \text{Hyp}(P) = \{ \sigma \in \text{Conj}(P); \mu_0 < \sigma < \mu_\land \} \), is the set of hypotheses of \( P \)
- \( \text{Sp}(P) = \{ \sigma \in \text{Conj}(P); \mu_\land \text{ NC } \sigma \} \), is the set of speculations of \( P \),
  where NC means non-comparable under \( \leq \).

verifying

\[
\text{Cons}(P) \cap \text{Hyp}(P) = \text{Cons}(P) \cap \text{Sp}(P) = \text{Hyp}(P) \cap \text{Sp}(P) = \emptyset.
\]

Notice that

- **Consequences follows** (in the partial order \( \leq \)) from \( \mu_\land \): all the premises explain the consequences
- **Hypotheses explain all the premises**, since \( \sigma < \mu_\land \leq \mu_i \), all the premises follow from each hypothesis.
- Speculations, are the conjectures that are neither \( \mu_\land \leq \sigma \) nor \( \sigma < \mu_\land \)
3.1. WHAT DOES IT MEAN “LOGIC”? 123

With all that, processes,

- $P \vdash \sigma$, with $\sigma \in \text{Conj}(P)$, is a guessing, or conjectural reasoning

- $P \vdash \sigma$, with $\sigma \in \text{Cons}(P)$, is a deduction, or deductive reasoning

- $P \vdash \sigma$, with $\sigma \in \text{Hyp}(P)$, is an abduction, or abductive reasoning

- $P \vdash \sigma$, with $\sigma \in \text{Sp}(P)$, is an speculation, or speculative reasoning

Concerning this types of conclusive reasonings, it should be pointed out what follows.

1. The set $\text{Conj}(P)$ is not always in $A = \mathbb{P}_0(X)$. Hence, it can’t be taken as a set of premises, it has not sense to consider $\text{Conj}(\text{Conj}(P))$, or $\text{Cons}(\text{Conj}(P))$.

2. If $P \subset Q$, it is $\text{Conj}(Q) \subset \text{Conj}(P)$, because if $\sigma \in \text{Conj}(Q)$, from $\text{Inf}Q \leq \text{Inf}P$, if $\text{Inf}Q \not\leq \sigma'$, it is $\text{Inf}P \not\leq \sigma'$. Hence, if $\sigma \in \text{Conj}(Q)$ it is $\sigma \in \text{Conj}(P)$.

3. The sets $\text{Hyp}(P)$ and $\text{Sp}(P)$ are not always in $\mathbb{P}_0(X)$. Hence, they can’t be taken as sets of premises.

4. If $P \subset Q$, it is $\text{Hyp}(Q) \subset \text{Hyp}(P)$, since $\sigma \in \text{Hyp}(Q), \text{or } \mu_0 < \sigma < \text{Inf}Q$ implies $\mu_0 < \sigma < \text{Inf}P$. Hence, the operators Conj and Hyp are anti-monotonic.

5. Concerning the operator Sp, if $P \subset Q$ it can be $\text{Sp}(P) \not\subseteq \text{Sp}(Q)$, and $\text{Sp}(Q) \not\subseteq \text{Sp}(P)$. Hence, Sp is a non-monotonic operator.

Remark 3.1.5. The operator Cons is greater than the operator $C_{<}$.

Proof. If $\sigma \in C_{<}(P)$, it is $\mu_i \leq \sigma$ from some $\mu_i \in P$. Since $\mu, \mu_i$, from all $\mu_i \in P$, it results $\mu \leq \sigma$, or $\sigma \in \text{Cons}(P)$. Hence, $C_{<}(P) \subset \text{Cons}(P)$, for all $P \in \mathbb{P}_0(X)$  □
3.2 Reasoning with conditionals: representation

3.2.1 What is it a conditional?

A conditional is an statement of the form 'If $a$, then $b$: $a \rightarrow b$, with two previous statements $a, b$. For example, “If it is raining, then I take an umbrella”, where $a = $ It is raining, $b = $ I take an umbrella, or “If the food is well cooked and well served, and the vine is of a good quality, then the tip will be higher than usual”, wit $a = $ The food is well cooked and well served, and the vine is of a good quality, $b = $ The tip will be higher than usual.

Notice that the first example is a crisp conditional, but the second is an imprecise one. In what follows we will take into account the representation of imprecise conditionals of the type

If $x$ is $P$, then $y$ is $Q$,

where $x \in X$, $y \in Y$, $P$ is a predicate (precise or imprecise) in $X$, and $Q$ is a predicate (precise or imprecise) in $Y$. For example, with $X = [0, 1]$, $Y = [0, 10]$,

If $x$ is small, then $y$ is big,

or,

If $x$ is small and $y$ is big, then $z$ is not small,

also with $x, y \in [0, 1]$ and $z \in [0, 10]$.

What it means to represent a conditional statement like “If $x$ is $P$, then $y$ is $Q$”? It mean to translate it in fuzzy terms. For example, ‘$x$ is $P$’ and ‘$y$ is $Q$’ will be translated by $\mu_P(x)$ and $\mu_Q(y)$ with adequate fuzzy sets $\mu_P, \mu_Q \in [0, 1]^X$, adequate in the sense that they collect the actual use of $P$ on $X$ and $Q$ on $Y$. 
3.2. REASONING WITH CONDITIONALS: REPRESENTATION

But how to represent the full statement “If $x$ is $P$, then $y$ is $Q$” := $\mu_P(x) \rightarrow \mu_Q(y)$? It is always done, in fuzzy logic, by means of a function $J : [0, 1] \times [0, 1] \rightarrow [0, 1]$, such that

$$\mu_P(x) \rightarrow \mu_Q(y) = J(\mu_P(x), \mu_Q(y)) \in [0, 1]$$

for all $x \in X$, and $y \in Y$. But, which function $J$ can be taken? It depends on the ‘meaning’ of the conditional statement, and this requires to look at what happens in general.

**Remark 3.2.1.** There are imprecise conditionals that is better not to transform in precise ones, for instance,

- If the turn is not so far, and the car’s speed is not high, then press softly the break,

- If the the food was well-cooked and of quality, and the service was good, the the tip should be higher of the 15%.

Both in common life and in technology, there are a lot of imprecise conditionals (rules). The need for its representation will be obvious in the next section.

3.2.2

Let us consider, in the first place, the case in which the statements are crisp and belong to a boolean algebra $(B, \cdot, +, \neg; 0, 1)$, where $\cdot, +, \neg$ stand, respectively, for the intersection (and), the union (or), and the complement (negation, not), 0 is the minimum element, and 1 is the maximum. As it is well known, this boolean algebra is naturally ordered by means of the partial order

$$a \leq b \iff a \cdot b = a \iff a + b = b \iff b = a + a' \cdot b.$$

Representing ‘If $a$, then $b$’ by $a \rightarrow b$, what is important is that from the set of premises $P = \{a, a \rightarrow b\}$ should follow the statement $b$ as a logical
CHAPTER 3. REASONING AND APPROXIMATE REASONING

consequence. For this goal, it should be \( a \cdot (a \rightarrow b) \neq 0 \), and \( a \cdot (a \rightarrow b) \leq b \).
In a boolean algebra it is

\[ a \cdot z \leq b \iff z \leq a' + b, \]

because of:

1. \( a \cdot z \leq b \Rightarrow a' + a \cdot z = a' + z \leq a' + b \), and since \( z \leq a' + z \), follows \( z \leq a' + b \)

2. \( z \leq a' + b \Rightarrow a \cdot z \leq a \cdot (a' + b) = a \cdot b \leq b \).

Then, \( a \cdot (a \rightarrow b) \leq b \iff a \rightarrow b \leq a' + b \).

A conditional function is a mapping \( \rightarrow : B \times B \rightarrow B \), such that \( a \cdot (a \rightarrow b) \leq b \) for all \( a, b \in B \). Hence, in a boolean algebra, the biggest conditional is \( a' + b \), the so-called material conditional, and any smaller function is also a conditional. For example, from

\[ a \cdot b \leq b \leq a' + b, \quad a' \leq a' + b \]

it follows that \( a \rightarrow b = a \cdot b, a \rightarrow b = b, a \rightarrow b = a' \), are conditionals. Analogously, from

\[ a' \cdot b' + a \cdot b \leq a' + b, \]

it also follows that \( a \rightarrow b = a' \cdot b' + a \cdot b \) is a conditional. Different ways of writing \( a' + b \) in a boolean algebra, are

\[ a' \cdot (b + b') + a \cdot b, \quad a' + a \cdot b, \quad b + a' \cdot b' \]

since \( a' \cdot (b + b') + a \cdot b = a' + a \cdot b = (a + a') \cdot (a' + b) = a' + b \), and \( b + a' \cdot b' = (b + a') \cdot (b + b') = a' + b \).

Notice that from \( z_1 \leq a' + b, z_2 \leq a' + b \), follows \( z_1 \cdot z_2 \leq (a' + b) \cdot (a' + b) = a' + b, z_1 + z_2 \leq (a' + b) + (a' + b) = a' + b \), hence, the union and the intersection of conditionals is also a conditional. For example, \( a \cdot b + a' + b = a' + b \) is obviously a conditional.
The two-variables functions \( a \rightarrow b \) such that \( a \rightarrow b \leq a' + b \), are not always expressible as a single formula with the connectives \( ' \), \( \cdot \), and \( + \) as the before considered cases. For example,

\[
a \rightarrow b = \begin{cases} 
a \cdot b, & \text{if } a \cdot b \neq 0 \\
a' + b, & \text{if } a \cdot b = 0,
\end{cases}
\]

verifies \((a \rightarrow b)a = \begin{cases} 
a \cdot b, & \text{if } a \cdot b \neq 0 \\
a \cdot (a' + b) = a \cdot b, & \text{if } a \cdot b = 0,
\end{cases} = a \cdot b \leq b\), that is, is a conditional. Analogously,

\[
a \rightarrow b = \begin{cases} 
1, & \text{if } a \leq b \\
b, & \text{otherwise},
\end{cases}
\]

verifies \((a \rightarrow b)a = \begin{cases} 
a, & \text{if } a \leq b \\
a \cdot b, & \text{otherwise},
\end{cases} \leq b\), that is, is a conditional.

**Remark 3.2.2.** If the boolean algebra \( B \) is complete, that is, for any \( A \subset B \), \( A \neq \emptyset \), it exists \( \text{Sup}A \in B \), then

\[
\text{Sup}\{z \in B; a \cdot z \leq b\} = \text{Sup}\{z \in B; z \leq a' + b\} = a' + b.
\]

**Remark 3.2.3.** The character of conditional of \( a' + b \) is exclusive of boolean algebras. That is, in any ortholattice, the validity of \( a \cdot (a' + b) \leq b \), for all \( a, b \), forces the ortholattice to be a boolean algebra.

**Remark 3.2.4.** \( a \rightarrow b \leq a' + b \), is a property that only holds in boolean algebras, that is, in ortholattices the equivalence \( a \cdot z \leq b \leftrightarrow z \leq a' + b \), is not valid. It only holds in boolean algebras. For example, in orthomodular lattices, both \( a \rightarrow_1 b = a' + a \cdot b \), and \( a \rightarrow_2 b = \tilde{b}' + a' \cdot \tilde{b}' \) (that verify \( a \rightarrow_2 b = \tilde{b}' \rightarrow_1 a' \)), are conditionals, but is neither \( a \rightarrow_1 b \leq a \rightarrow_2 b \) nor \( a \rightarrow_2 b \leq a \rightarrow_1 b \).

The conditional \( a \rightarrow_1 b = a' + a \cdot b \) is called the Sasaki hook, and \( a \rightarrow_2 b = \tilde{b}' + a' \cdot \tilde{b}' \) is the Dishkant hook, and, of course, only in boolean algebras are both coincidental with \( a' + b \). The Sasaki and the Dishkant hooks are
used as models for the conditional statements in the reasoning in Quantum Logic.

**Remark 3.2.5.** The scheme of *Modus Ponens* corresponds to *forwards reasoning*, that is, goes from the antecedent $a$ to the consequent $b$ thanks to the conditional $a \rightarrow b$. *Backwards reasoning* goes from the consequent to the antecedent (also thanks to $a \rightarrow b$), ad it is modeled by the *Modus Tollens* scheme.

\[
\begin{array}{c}
\text{If } a, \quad \text{then } b \\
\text{not } b
\end{array}
\]

that is translated by $\theta' \cdot (a \rightarrow b) \leq a' \Leftrightarrow a \rightarrow b \leq (\theta')' + a' = a' + b$. Thus, in boolean algebras, $a \rightarrow b = a' + b$, allows backwards reasoning, provided $\theta' \cdot (a \rightarrow b) = \theta' \cdot (a' + b) = a' \cdot \theta' \neq 0$, or $a + b 
eq 1$. Nevertheless, although $\theta' \cdot (a \cdot b) = 0 \leq a'$, it is clear that the conjunctive conditional $a \rightarrow b = a \cdot b$ does not allow backwards reasoning since $\theta' \cdot (a \rightarrow b) = 0$.

### 3.2.3

Let us return to the case of fuzzy logic, that is, to a conditional linguistic expression, or rule, like ‘If $x$ is $P$, then $Y$ is $Q$’, represented in fuzzy terms by

\[ (\mu_P \rightarrow \mu_Q)(x, y) = J(\mu_P(x), \mu_Q(y)), \]

for all $x \in X, y \in Y$. The problem is which function $J : [0, 1] \times [0, 1] \rightarrow [0, 1]$, is to be taken at each case if, of course, it gives a conditional. That is, if from the premises \{x is P, If x is P, then y is Q\} follows ‘y is Q’ as a logical consequence. Formally speaking, it should exist a continous t-norm $T_0$ such that

\[ T_0(\mu_P(x), J(\mu_P(x), \mu_Q(y))) \leq \mu_Q(y) \]

for all $x \in X, y \in Y$. This condition, that should hold for any $\mu_P(x) \in [0, 1]$, and any $\mu_Q(y) \in [0, 1]$, conducts to the inequality

\[ T_0(a, J(a, b)) \leq b \]
for all $a, b$ in $[0, 1]$. $J$ should verify to represent conditional statements. It is called the Modus Ponens Inequality, since it allows the scheme of reasoning

\[
\begin{array}{c}
\text{If } x \text{ is } P, \text{ then } y \text{ is } Q \\
x \text{ is } P \\
y \text{ is } Q
\end{array}
\]

called the scheme of Modus Ponens. $J$ is called a $T_0$-conditional.

There is a theorem, whose proof will be omitted, showing that being $T_0$ a continuous t-norm, it is

\[
T_0(a, J(a, b)) \leq b \iff J(a, b) \leq J_{T_0}(a, b) = \sup\{ z \in [0, 1]; T_0(z, a) \leq b \}.
\]

Hence, for each $T_0$, the greatest $T_0$-conditional is the function $J_{T_0}$, since, it verifies the MP-inequality $T_0(a, J_{T_0}(a, b)) = \min(a, b) \leq b$.

**Remark 3.2.6.** For reasons that will be latter on presented, $T$-conditionals $J_T$ are called R-implications (R shorting residuated). They come directly from the boolean equation $a' + b = \sup\{ z; a \cdot z \leq b \}$.

If $\mu, \sigma \in \{0, 1\}^X$, take $(\mu \rightarrow \sigma)(x, y) = J_T(\mu(x), \sigma(y)) = \sup\{ z \in [0, 1]; T(z, \mu(x)) \leq \sigma(y) \}$. If $\mu(x) \in \{0, 1\}, \sigma(y) \in \{0, 1\}$, it is

\[
J_T(\mu(x), \sigma(y)) = \begin{cases}
J_T(0, 0) = 1 \\
J_T(0, 1) = 1 \\
J_T(1, 0) = 0 \\
J_T(1, 1) = 1,
\end{cases}
\]

that coincides with the values of $\max(1 - \mu(x), \sigma(y))$. That is, all R-implications do coincide with the boolean material conditional $\mu' + \sigma$ in the case that $\mu$ and $\sigma$ are crisp sets. R-implications generalize the material conditional. Of course, this will happen with any $J$ such that

\[
J(0, 0) = 1, J(0, 1) = 1, J(1, 0) = 0, J(1, 1) = 1
\]
Example 3.2.7. The immediate generalization of the boolean conditional \( a \rightarrow b = a' + b \) is given by \( (\mu' + \sigma)(x, y) = \mu'(x) + \sigma(y) \), that is, by \( J(a, b) = S(N(a), b) \), for all \( a, b \) in \([0, 1] \). These operators are called S-implications (S shortens 'strong'). With,

- \( S = \text{max} \), \( N = N_0 \), is \( J(a, b) = \text{max}(1 - a, b) \), called the Kleene-Diennes conditional.
- \( S = \text{prod}^* \), \( N = N_0 \), is \( J(a, b) = 1 - a + ab \), called the Reichenbach conditional
- \( S = W^* \), \( N = N_0 \), is \( J(a, b) = \text{min}(1 - a + b) \), called the Lukasiewicz conditional

Notice that the MP inequality \( T_0(S(N(a), b) \leq a \), is verified on the last three cases with \( T_0 = W \):

- \( W(a, \text{max}(1 - a, b)) = \text{max}(0, a + b - 1) = W(a, b) \leq b \)
- \( W(a, 1 - a + a \cdot b) = \text{max}(0, a \cdot b) = a \cdot b \leq b \)
- \( W(a, \text{min}(1, 1 - a + b)) = \text{max}(0, \text{min}(a, b)) = \text{min}(a, b) \leq b \)

hence, the three cases are W-conditionals.

Example 3.2.8. Let us see how is \( J_T \), when \( T \) is, respectively, the continuos t-norms \( \text{min} \), \( \text{prod}_\varphi \), \( W_\varphi \).

- \( T = \text{min} \), \( J_{\text{min}}(a, b) = \sup \{ z \in [0, 1]; \text{min}(z,a) \leq b \} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases} \) (Gdel implication)
- \( T = \text{prod}_\varphi \), \( J_T(a, b) = \sup \{ z \in [0, 1]; \varphi(a) \cdot \varphi(z) \leq \varphi(b) \} = \begin{cases} 1, & \text{if } a \leq b \\ \varphi^{-1}(\frac{\varphi(b)}{\varphi(a)}), & \text{if } a > b \end{cases} \) (Goguen implication)
- \( T = W_\varphi \), \( J_T(a, b) = \sup \{ z \in [0, 1]; \varphi^{-1}(W(\varphi(a), \varphi(z)) \leq b) = \varphi^{-1}(\text{min}(1, 1 - \varphi(a) + \varphi(b))) \) (Lukasiewicz implications).
Since each $J_T$ is a $T$-conditional, Gdel’s is a min-conditional, Goguen’s are prod-$\varphi$-conditionals, and Lukasiewicz’s are $W_\varphi$-conditionals. Notice that the S-implication of the form

$$W^*_\varphi(N_\varphi(a), b) = \varphi^{-1}(W^*(\varphi(N_\varphi(a)), \varphi(b))) = \varphi^{-1}(W^*(1 - \varphi(a), \varphi(b))) = \varphi^{-1}(\min(1, 1 - \varphi(a) + \varphi(b)))$$

are exactly the Lukasiewicz’s R-implications: the only R-implications that are S-implications are the Lukasiewicz’s ones. If it were

$$J_{\min}(a, b) = S(N(a), b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$$

it will result

$$S(a, b) = \begin{cases} 1, & \text{if } N(a) \leq b \\ b, & \text{if } N(a) > b \end{cases}$$

a function that is not a t-conorm, since $S(a, 0) = 0 \neq a$, if $a > 0$. Hence, $J_{\min}$ is not an S-implication. An analogous reasoning shows that $J_{\prod_\varphi}$ are not S-implications.

**Example 3.2.9.** The protoform $\mu \rightarrow \sigma = \mu' + \mu \cdot \sigma$ (coming from the Sasaki hook), gives

$$J_1(a, b) = S(N(a), T(a, b)),$$

with $S$ a continuous t-conorm, $T$ a continuous t-norm, an $N$ an strong negation. These function are called $Q$-conditionals ($Q$ for Quantum). For example,

- $S = \max$, $T = \min$, $N = N_0$, is $J_1(a, b) = \max(1 - a, \min(a, b))$, is the so-called Early-Zadeh operator
- $S = \max$, $T = \prod$, $N = N_0$, is $J_1(a, b) = \max(1 - a, ab)$
- $S = \prod^*$, $T = \min$, $N = N_0$, is $J_1(a, b) = 1 - a + a^2b$
\[ S = W^*, \quad T = W, \quad N = N_0, \text{ is } J_1(a, b) = \max(1 - a, b), \text{ that coincides with the Kleene-Dienes implication} \]

\[ S = W^*, \quad T = \text{prod}, \quad N = N_0, \text{ is } J_1(a, b) = 1 - a + ab, \text{ that coincides with the Reichenbach implication} \]

\[ S = W^*, \quad T = \text{min}, \quad N = N_0, \text{ is } J_1(a, b) = \min(1, 1 - a + b), \text{ that coincides with the Łukasiewicz implication} \]

With which t-norm \( T_0 \) do verify the MP inequality these Q-operators? For instance,

\[ W(a, \max(1-a, \min(a, b))) = \max(0, a+\min(a, b) - 1) = W(a, \min(a, b)) \leq \min(a, b) \leq b \]

\[ W(a, \max(1 - a, a \cdot b)) = \max(0, a + a \cdot b) - 1) = W(a, a \cdot b) \leq a \cdot b \leq b \]

\[ W(a, 1 - a + a^2 b) = \max(0, a^2, b) = a^2 b \leq b \]

\[ W(a, \max(1 - a, b)) \leq b \text{ (as it is proven before)} \]

\[ W(a, 1 - a + ab) \leq b \text{ (as it is proven before)} \]

**Example 3.2.10.** The protoform \( \mu \rightarrow \sigma = \sigma + \mu \cdot \sigma' \) (coming from the Dishkant hook), gives the D-operators:

\[ J_2(a, b) = S(b, T(N(a), N(b))), \]

with which \( J_2(N(b), N(a)) = S(N(a), T(a, b)) = J_1(a, b) \) or, equivalently, \( J_2(a, b) = J_1(N(b), N(a)) \): D-operators are the contrasymmetricals of Q-operators. Hence, it can be repeated all that has been said for \( J_1 \). For example,

If \( S = \max, \quad T = \min, \quad N = N_0, \) it is \( J_2(a, b) = J_1(1 - a, 1 - b) = \max(b, \min(1 - b, 1 - a)) = \max(b, 1 - \max(a, b)), \) that verifies

\[ W(a, \max(b, 1 - \max(a, b))) = \max(0, \max(a + b - 1, a - \max(a, b))) = \]
\[ \begin{cases} W(a, b), & \text{if } b \leq a \text{ or } b > a \text{ and } b > \frac{1}{2} \\ 0, & \text{if } b > a \text{ and } b \leq \frac{1}{2} \end{cases} \leq b. \text{ It is a W-conditional.} \]

**Example 3.2.11.** The protoform \( \mu \to \sigma = \mu \cdot \sigma \) (coming from the classical conjunctive conditional \( a \to b = a \cdot b \)), gives

\[ J(a, b) = T(a, b), \]

functions with the inconvenience of the property \( J(a, b) = J(b, a) \), but verifying,

\[ T_0(a, J(a, b)) = T_0(a, T(a, b)) \leq \min(a, \min(a, b)) = \min(\min(a, a), b) = \]

\[ = \min(a, b) \leq b \]

that is, all of them are conditionals for any t-norm \( T_0 \) and in particular, for the greatest of them. Always are taken as min-conditionals. For example,

- If \( T = \min \), \( J(a, b) = \min(a, b) \), is called the Mamdani conditional
- If \( T = \text{prod}_\varphi \), \( J(a, b) = \varphi^{-1}(\varphi(a) \cdot \varphi(b)) \), are called Larsen conditionals
- \( T = W_\varphi \) is never used, since it can be \( J(a, b) = 0 \) with \( a > 0 \) and \( b > 0 \).

For example, with \( \varphi(x) = \frac{x(1+x)}{2} \) (an order automorphism), it is \( \varphi^{-1}(x) = \frac{\sqrt{x^2 + T - 1}}{2} \), and

\[ J(a, b) = \varphi^{-1}(\varphi(a) \cdot \varphi(b)) = \varphi^{-1}\left(\frac{a(1+a)}{2} \cdot \frac{b(1+b)}{2}\right) = \varphi^{-1}\left(\frac{ab(1+a)(1+b)}{4}\right) = \]

\[ = \frac{\sqrt{ab(1+a)(1+b) + 1} - 1}{2} \]

that, of course, is a min-conditional.
Remark 3.2.12. A way of avoiding the undesirable symmetry \( J(a, b) = J(b, a) \) in the case of Mamdani-Larsen min-conditionals, is taking

\[
J(a, b) = T(a^r, b^s),
\]

with real numbers \( r, s \) with \( 1 > s \). Then \( T_0(a, T(a^r, b^s)) \leq \min(\min(a^r, b^s), b^s) \leq b^s \leq b \), and \( J \) is a min-conditional.

Example 3.2.13. Once given a conditional statement ‘If \( x \) is \( P \), then \( y \) is \( Q \)’, and represented ‘\( x \) is \( P \)’ by \( \mu_P(x) \), and ‘\( y \) is \( Q \)’ by \( \mu_Q(y) \), it remains to be understood what it is meant by the ‘statement’ \( \mu_P(x) \rightarrow \mu_Q(y) \). It could be, or not to be, \( \mu_P(x) \rightarrow \mu_Q(y) = (\mu_P \rightarrow \mu_Q)(x, y) \), with \( \mu_P \rightarrow \mu_Q \) a fuzzy set in \( X \times Y \), identified with some expression involving the connectives and (\( \cdot \)), or (+), not (\( \lnot \)). In the affirmative case, it is said that \( \mu_P \rightarrow \mu_Q \) is expressible in material form, for example, \( \mu_P \rightarrow \mu_Q = \mu'_P + \mu_Q \), or \( \mu_P \rightarrow \mu_Q = \mu'_P + \mu_P \mu_Q \), etc. These material forms are called protoforms.

If \( \mu_P \rightarrow \mu_Q \) does not correspond with a protoform, one can try to represent it by means of an R-implication, that is, by \( J_{\text{min}} \) or by some \( J_{\text{prod}, \varphi} \), since all the \( J_{\varphi} \) do correspond to a protoform \( \mu'_P + \mu_Q \), with \( + \) represented by \( W_\varphi^* \) and \( \lnot \) by \( N_\varphi \).

In addition, there is a problem that should be taken into account when representing \( \mu_P \rightarrow \mu_Q \). The problem is the following. Suppose that we know \( \mu_P \rightarrow \mu_Q \) should be represented by a function \( J \) that is a min-conditional, but that we are not able to decide a protoform and we take \( J_{\text{min}} \). Since \( J \leq J_{\text{min}} \), we will reach the biggest possible output. This should be known. Analogously, if \( J \) should be a prod-conditional, from \( J \leq J_{\text{prod}} \), follows the same comment.

Example 3.2.14. Let’s stop for a while in the above mentioned concept of implication function, a concept that comes directly from the properties shown by the boolean conditional \( a \rightarrow b = a' + b \), whose truth value is usually represented by

\[
v(a \rightarrow b) = \max(1 - v(a), v(b)).
\]
3.2. **REASONING WITH CONDITIONALS: REPRESENTATION**

Look that, if $b_1 \leq b_2$, it follows $a' + b_1 \leq a' + b_2$, or $a \rightarrow b_1 \leq a \rightarrow b_2$, if $a_1 \leq a_2$, is $a_2' \leq a_1'$, and $a_2 \rightarrow b \leq a_1 \rightarrow b$. Since $v(a)$ and $v(b)$ only take the values \{0, 1\}, $v(a \rightarrow b)$ has the truth-table

<table>
<thead>
<tr>
<th>$v(a)$</th>
<th>$v(b)$</th>
<th>$v(a \rightarrow b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Because of that, a function $J : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a fuzzy implication function provided:

1. $J$ is decreasing in its first variable, and non-decreasing in its second one

2. $J(0, 0) = J(0, 1) = J(1, 1) = 1$, $J(1, 0) = 0$

Obviously, S-implications and R-implications are fuzzy implication functions, but Q and D operators are not always so, since, for instance

$$J_S(0.4, 0.3) = \max(1 - 0.4, \min(0.4, 0.3)) = 0.6$$

$$J_D(0.3, 0.1) = \max(0.1, \min(0.7, 0.9)) = 0.7$$

$$J_D(0.3, 0.4) = \max(0.4, \min(0.7, 0.6)) = 0.6,$$

that is, for instance $0.1 < 0.4$ but $J_D(0.3, 0.1) > J_D(0.3, 0.4)$.

Analogously, $J(a, b) = T(a, b)$ are not implication functions, since $J(0, 0) = T(0, 0) = 0$, and $a_1 \leq a_2 \Rightarrow J(a_1, b) \leq J(a_2, b)$. Also Sasaki operators are not, for example, $0.4 < 0.8$, but $J_S(0.4, 0.9) = \max(1 - 0.4, \min(0.4, 0.9)) = \max(0.6, 0.4) = 0.6 < 0.8 = J_S(0.8, 0.9)$, that is, $J_S$ is non-decreasing in the second variable.

To represent conditional statements as they appear in language, the concept of implication function is sometimes excessive. What is needed are just T-conditionals, except in the cases where more properties are necessary to represent the meaning of the conditional statements.
Remark 3.2.15. There is a question that is not independent or the representation $J$ for the conditional statement. The question is: What are we going to do with $J$? Which is the purpose for using $J$?

We are to make an inference that, in principle, could be forwards

$$\{\mu, \mu \rightarrow \sigma\} \vdash \sigma,$$

or backwards,

$$\{\sigma', \mu \rightarrow \sigma\} \vdash \mu'.$$

The first type of inference corresponds to search for the solutions of $\mu \cdot (\mu \rightarrow \sigma) \leq \sigma$, that is, for $J$ and $T_0$ such that,

$$T_0(\mu(x), J(\mu(x), \sigma(y))) \leq \sigma(y); \quad \forall x, y \in X. \quad (\ast)$$

The second type corresponds to search for the solutions of $\sigma' \cdot (\mu \rightarrow \sigma) \leq \mu'$, that is, for $J$, $T_0$ and $N$ such that,

$$T_0(N(\sigma(y)), J(\mu(x), \sigma(y))) \leq N(\mu(x)); \quad \forall x, y \in X. \quad (***)$$

Hence, given $J$, we need to know $T_1$ such that $T_1(a, J(a, b)) \leq b$, for forward inference, and given $J$ and $N$, we need to know $T_0$ such that $T_0(N(b), J(a, b)) \leq N(a)$, for backwards inference. Notice that the two t-norms $T_0$ in ($\ast$) and (***) do not be necessary coincidental. For example, given $J(a, b) = \max(1 - a, b)$, with $N = N_0$, can we do backwards inference? To answer this question we just need to know if there is a continuous t-norm $T_0$ such that $T_0(1 - b, \max(0, \max(1 - a, b))) \leq 1 - a$, with $a = 1$ it results $T_0(1 - b, b) = 0$, and $T_0 = W$. Then, since,

$$W(1 - b, \max(1 - a, b)) = \max(0, 1 - b + \max(1 - a, b) - 1) = \max(0, \max(1 - a - b, 0)) = \begin{cases} 1 - a - b, & \text{if } a + b \leq 1 \\ 0, & \text{if } a + b > 1 \end{cases} \leq 1 - a,$$

because $b \geq 0$ implies $-b \leq 0$, and $1 - a - b \leq 1 - a$. Finally, the answer is: Yes, with $T_0 = W$. It can also be done backwards inference in the following cases,

1. With $J_{\min}$, since $W(1 - b, J_{\min}(a, b)) = \begin{cases} 1 - b, & \text{if } a \leq b \\ W(1 - b, b) = 0, & \text{if } a > b \end{cases} \leq 1 - a$
2. With $J_{\text{prod}}$, since $W(1-b, J_{\text{prod}}(a, b)) = \begin{cases} 1-b, & \text{if } a \leq b \\ \frac{1-b}{a}, & \text{if } a > b \end{cases} \leq 1-a$

3. With $J_W$, since $W(1-b, \min(1, 1-a + b)) = \min(1-b, 1-a) \leq 1-a$

4. With $J(a, b) = 1-a + ab$, since $W(1-b, \min(1, 1-a + ab)) = (1-b)(1-a) \leq 1-a$

5. With $J(a, b) = \max(1-a, \min(a, b))$, since $W(1-b, \max(1-a, \min(a, b))) \leq W(1-b, \max(1-a, b)) \leq 1-a$.

6. With $J(a, b) = \text{prod}^*(1-a, a \cdot b) = 1-a + a^2 b$, since $W(1-b, \text{prod}^*(1-a, a \cdot b)) \leq W(1-b, \text{prod}^*(1-a, b)) = W(1-b, 1-a + a \cdot b) \leq 1-a$

Nevertheless, the case $J(a, b) = T(a, b)$ is, actually, negative. To have $T_1(1-b, T(a, b)) \leq 1-a$, it is necessary (take $a = 1$) that $T_1(1-b, b) = 0$, that is, $T_1 = W$, but it is $W(1-b, T(a, b)) = \max(0, T(a, b) - b) = 0$, since $T(a, b) \leq b$ implies $T(a, b) - b \leq 0$.

Hence, by one side from $W(1-b, T(a, b)) = 0 \leq 1-a$, it seems that backwards inference is possible. But, given the scheme: “$\sigma', \mu \rightarrow \sigma : \mu'$”, what results is $\sigma' \cdot (\mu \rightarrow \sigma) = \mu_0$, that forces $\text{Conj}(\{\sigma', \mu \rightarrow \sigma\}) = \emptyset$. In conclusion, Mamdani-Larsen conditionals don’t allow backwards inference.

Remark 3.2.16. It should be pointed out that except $J_{\text{min}}, J_{\text{prod}}$, and $J(a, b) = T(a, b)$, most of the functions $J$ are $T_0-$conditionals for $T_0 = W$, and almost all do also verify backwards inference also with $T_0 = W$. And the t-norms in the Lukasiewicz’s family show the disturbing problem of having zero-divisors!

Remark 3.2.17. The name R-implication, or residuated implication, comes from the idea of ‘residuum’ that clearly appear in the case of $J_{\text{prod}}$ when

If $a > b$, then $J_{\text{prod}}(a, b) = \frac{b}{a}$.

Remark 3.2.18. In the same vein under which it was proven that R-implications $J$ with $T \neq W$ are not S-implications, it is easy to show that they are not
expressible in material protoform, that is, by an expression with logical connectives. Take the perhaps more general material protoform $\mu^l \cdot (\sigma + \sigma') + \mu \cdot \sigma$, is it possible that

$$J_{T_0}(a, b) = S_1(T_1(N_1(a), S_2(b, N_2(b)), T_2(a, b))),$$

for $T_0 \neq W_\varphi, S_1$ and $S_2$ continuous t-conoms, $T_1, T_2$ continuous t-norms, and $N_1, N_2$ strong negations?

Whit $b = 0$, it follows

$$J_{T_1}(a, 0) = \sup\{z \in [0, 1]; T(z, a) \leq 0\} = 0$$

$$S_1(T_1(N_1(a), S_2(0, 1)), 0) = T_1(N_1(a), 1) = N_1(a),$$

or, $S_1(N_1(a), 0) = 0$. This means $S_1$ is not a t-conorm. Hence, the decision of representing an R-implication can’t be taken from a material protoform interpretation of it.

### 3.3 Short note on other modes of reasoning

The mode of reasoning given by the scheme

$$\{\mu, \mu \to \sigma\} \vdash \sigma$$

is classically called *Modus Ponendo Ponens* (from the Latin, mode of starting the truth (of $\sigma$) by placing the truth (of $\mu$)), or, for short *Modus Ponens*. That given by the scheme $\{\sigma', \mu \to \sigma\} \vdash \mu'$ is classically called *Modus Tollendo Tollens* (from the Latin, mode of stating the falsity (of $\mu$) by placing the falsity (of $\sigma$)), or, for short *Modus Tollens*. They correspond to what we called forwards and backwards reasoning. But there are again other modes of reasoning that can be considered, for example,

- *Modus Tollendo Ponens*, given by the scheme $\{\mu', \mu + \sigma\} \vdash \sigma$ and also called Mode of Disjunctive Reasoning, and classically proven by $\mu' \cdot (\mu + \sigma) = \mu \cdot \mu' + \mu' \cdot \sigma = \mu' \cdot \sigma \leq \mu$ (in a boolean algebra).
• **Modus Ponendo Tollens**, given by the scheme \( \{ \mu, (\mu \cdot \sigma)' \} \vdash \sigma' \), classically proven by \( \mu \cdot (\mu \cdot \sigma)' = \mu \cdot (\mu' + \sigma') = \mu \cdot \sigma' \leq \sigma' \) (in a boolean algebra).

• **Constructive Dilemma**, given by the scheme \( \{ \mu + \lambda, \mu \rightarrow \sigma, \lambda \rightarrow \eta \} \vdash \sigma + \eta \), classically proven by \( (\mu + \lambda) \cdot (\mu \rightarrow \sigma) \cdot (\lambda \rightarrow \eta) = (\mu + \lambda) \cdot (\mu' + \sigma) \cdot (\lambda' + \eta) = \mu \sigma \lambda + \eta \mu \sigma + \eta \lambda \sigma \leq \sigma + \eta \) (in a boolean algebra where \( a \rightarrow b = a' + b \)).

• **Destructive Dilemma**, given by the scheme \( \{ \mu' + \sigma', \lambda \rightarrow \mu, \eta \rightarrow \sigma, \} \vdash \lambda' + \sigma' \), classically proven by \( (\mu' + \sigma')(\lambda + \mu)(\eta + \sigma) = \lambda'(\mu' \cdot \eta' + \mu' \sigma) + \eta'(\sigma' \cdot \lambda' + \mu \cdot \sigma') \leq \lambda' + \eta' \) (in a boolean algebra where \( a \rightarrow b = a' + b \)).

What in the fuzzy case? For example, in the case of the Disjunctive Mode we need to find all the possibilities for \( \mu' \cdot (\mu + \sigma) \leq \sigma \), that is, to solve the functional equation

\[
T(N(a), S(a, b)) \leq b
\]

for all \( a, b \) in \([0, 1]\), in the three variables \( T, S, N \). With \( b = 0 \), it follows \( T(N(a), a) = 0 \), or \( T = W_\varphi, N \leq N_\varphi \). Taking \( N = N_\varphi \) it results

\[
W_\varphi(N_\varphi(a), S(a, b)) = \varphi^{-1}(\max(0, 1 - \varphi(a) + \varphi(S(a, b)) - 1))
\]

\[
= \varphi^{-1}(\max(0, +\varphi(S(a, b)) - \varphi(a))) \leq b, \text{ implying } \varphi(S(a, b)) - \varphi(a) \leq \varphi(b)
\]

or \( \varphi(S(a, b)) \leq \varphi(a) + \varphi(b) \). Hence, \( S(a, b) \leq \varphi^{-1}(\min(1, \varphi(a) + \varphi(b))) = W^*(a, b) \). For example, it can be taken \( S = W^* \) or \( S = \max \).

\[
W_\varphi(N_\varphi(a), \max(a, b)) = \varphi^{-1}(\max(0, \varphi(b) - \varphi(a))) \leq \varphi^{-1}(\varphi(b)) = b
\]

\[
W_\varphi(N_\varphi(a), W^*(a, b)) = \varphi^{-1}(\min(1 - \varphi(a), \varphi(b))) \leq \varphi^{-1}(\varphi(b)) = b
\]

Hence, the disjunctive mode can be used in, for example, the cases \((W_\varphi, N_\varphi, \max)\) and \((W_\varphi, N_\varphi, W^*_\varphi)\).

**Remark 3.3.1.** It should be pointed out that the Modus Ponendo Tollens (MPT) can be reduced, in the case of duality, to the disjunctive mode by means of the change \( \mu = \alpha', \sigma = \beta' \), in which case since \((\mu \cdot \sigma)' = \mu' + \sigma'\) it follows \( \mu \cdot (\mu \cdot \sigma)' = \mu \cdot (\mu' + \sigma') = \alpha' \cdot (\alpha + \beta) \leq \beta = \sigma' \). Hence, it holds with the triplet \((W_\varphi, N_\varphi, W^*_\varphi)\).
3.4 Inference with fuzzy rules

What is the problem? A central topic fuzzy logic deals with non-rigid, dynamic systems involving ‘variables’ $x_1, \ldots, x_n, y$ taking values in, respectively, universes $X_1, \ldots, X_n, Y$, and constrained by imprecise rules $r_i$ of the type

If $x_1$ is $P_{1i}$, and $x_2$ is $P_{2i}$, ..., and $x_n$ is $P_{ni}$, then $y$ is $Q_i$ ($1 \leq i \leq n$), with predicates $P_{ji}(1 \leq j \leq n)$ in $X_j$, and $Q_i$ in $Y$.

Let’s consider the simplest case with two variables $x \in X, y \in Y$, constrained by a single rule *If $x$ is $P$, then $y$ is $Q$.*

When observing the system $(x, y)$, the variable $x$ in the rule’s antecedent not always will show ‘$x$ is $P$’, but ‘$x$ is $P^*$’ with $P^*$ some predicate slightly modificate from $P$. For example, is $P = \text{short}$ it could be $P^* = \text{very short, almost short}$, etc. A concrete example is

Rule: If tomatoes are red, they are ripe

Observation: Tomatoes are very red

Conclusion: Tomatoes are very ripe,

where $P = \text{red}$, $Q = \text{ripe}$, $P^* = \text{very red}$, and $Q^* = \text{very ripe}$. Hence, the corresponding scheme of forwards reasoning is

Rule: If $x$ is $P$, then $y$ is $Q$

Observation: $x$ is $P^*$

Conclusion: $y$ is $Q^*$

where $P, Q, P^*$ are known, and $Q^*$ is unknown. This scheme is known as the *Generalized Modus Ponens* (GMP), and to find $Q^*$ through fuzzy artillery it is needed to have the representations

- $\mu_P \in [0, 1]^X$, of $P$.
- $\mu_Q \in [0, 1]^Y$, of $Q$. 

3.4. INFERENCE WITH FUZZY RULES

- $\mu_{P^*} \in [0, 1]^X$, of $P^*$.
- $(\mu_P \rightarrow \mu_Q)(x, y) = J(\mu_P(x), \mu_Q(y))$, with a convenient $T$-conditional $J$.

From this representations should follow a representation of $Q^*$, that is, $\mu_{Q^*} \in [0, 1]^Y$, by taking into account that it should be a logical consequence of the set of premises $\{\mu_{P^*}, \mu_P \rightarrow \mu_Q\}$: the so-called Generalized Modus Ponens (GMP). That is, $\mu_{Q^*}$ does verify

$$0 \neq T_0(\mu_{P^*}(x), J(\mu_P(x), \mu_Q(y))) \leq \mu_{Q^*}(y), \quad (3.1)$$

for all $x \in X, y \in Y$, and a continuous t-norm $T_0$ verifying

$$0 \neq T_0(\mu_P(x), J(\mu_P(x), \mu_Q(y))) \leq \mu_Q(y),$$

for all $x \in X, y \in Y$, stating that if $P^* = P$, then $Q^* = Q$. This is done by preserving the Modus Ponens (MP) in the occasion in which ‘$x$ is $P$’ is observed. The fuzzy set $\mu_{P^*}$ is called the input

![Diagram](image)

and $\mu_{Q^*}$ is the output.

Obviously,

$$\mu_{Q^*}(y) = \sup_{x \in X} T_0(\mu_{P^*}(x), J(\mu_P(x), \mu_Q(y))), \forall y \in Y,$$

is the greatest function verifying the GMP (3.1). This formula is known as the Compositional Rule of Fuzzy Inference (CRI, for short), and was introduced by Lotfi A. Zadeh as the output fuzzy logic considers in the systems that are
described by rules. It is not to forget that $T_0$ is the continuous t-norm that makes $J$ a $T_0$-conditional.

Sometimes, the input is just numerical, crisp in the form ‘$x = x_0$’, that is, ‘$x$ is $P$’ is ‘$x = x_0$’, or ‘$x \in \{x_0\}$’ and then $\mu_{P*} = \mu_{\{x_0\}}$, with

$$
\mu_{\{x_0\}}(x) = \begin{cases} 
1, & \text{if } x = x_0 \\
0, & \text{if } x \neq x_0.
\end{cases}
$$

In this case,

$$
\mu_{Q*}(y) = \sup_{x \in X} T_0(\mu_{\{x_0\}}(x), J(\mu_P(x), \mu_Q(y))) = J(\mu_P(x_0), \mu_Q(y)), \forall y \in Y,
$$
a simpler expression that does not force to compute the $\sup_{x \in X}$.

Sometimes, in addition to the input, the rule’s consequent $\mu_Q$ is also numerical, that is ‘$y$ is $Q$’ is ‘$y = y_0$’, or ‘$y$ is $y_0$’, or ‘$y \in \{y_0\}$’. In this case

$$
\mu_Q(y) = \mu_{\{y_0\}}(y) = \begin{cases} 
1, & \text{if } y = y_0 \\
0, & \text{if } y \neq y_0.
\end{cases}
$$

and the output is

$$
\mu_{Q*}(y) = J(\mu_P(x_0), \mu_{\{y_0\}}(y)) = \begin{cases} 
J(\mu_P(x_0), 1), & \text{if } y = y_0 \\
J(\mu_P(x_0), 0), & \text{if } y \neq y_0.
\end{cases}
$$

**Example 3.4.1.** Take $X = [0, 1], Y = [0, 10]$, and the rule ‘If $x$ is small, then $y$ is big’, with the observation that ‘$x$ is big’ and $J(a, b) = \max(1-a, b)$. With the membership functions $\mu_P(x) = 1 - x, \mu_Q(y) = \frac{y}{10}, \mu_{P*}(x) = x$, it results

$$
\mu_{Q*}(y) = \sup_{x \in [0, 1]} W(x, \max(x, \frac{y}{10})) = W(1, \max(1, \frac{y}{10})) = 1,
$$
or, $\mu_{Q*} = \mu_1$, that means $Q^* = \text{all}$.

**Example 3.4.2.** With the same rule of last example and the input $x_0 = 0.5$, it is

$$
\mu_{Q*}(y) = \max(0.5, \max(0.5, \frac{y}{10})), \forall y \in [0, 1]
$$
graphically,
Example 3.4.3. With the rule ‘If x is small, then $y = 8$’, and the input $x = 0.5$, is

$$\mu_{Q^*}(y) = \max(0.5, \mu_{[8]}(y)) = \begin{cases} 1, & \text{if } y = 8 \\ 0.5, & \text{if } y \neq 8, \end{cases}$$

graphically,

Remark 3.4.4. In general, it is not easy to assign a name to the functional output $\mu_{Q^*}$, that is, to express $Q^*$ linguistically. In the example 3.4.2, it could be said $Q^* = \text{big}$ after 5 and constantly equal to 0.5 before 5. In the example 3.4.3, it could be said $Q^* = \text{almost always 0.5}$.

3.4.1

Let’s consider the particular case where both universes $X$ and $Y$ are finite sets. If

$$X = \{x_1, \ldots, x_n\}, \quad Y = \{y_1, \ldots, y_m\},$$

fuzzy sets $\mu_P, \mu_Q$ and $\mu_{PS}$ are of the forms:

- $\mu_P = r_1/x_1 + \ldots + r_n/x_n$, meaning $\mu_P(x_i) = r_i, 1 \leq i \leq n$
• $\mu_Q = s_1/y_1 + \ldots + s_n/y_m$, meaning $\mu_Q(y_j) = s_i, 1 \leq j \leq m$

• $\mu_{P^*} = r_1^*/x_1 + \ldots + r_n^*/x_n$, meaning $\mu_{P^*}(x_i) = r_i^*, 1 \leq i \leq n$

Provided $J$ is an adequate $T_0$–conditional to represent the rule ‘If $x$ is $P$, then $y$ is $Q$’ ($(\mu_P \rightarrow \mu_Q) = J(\mu_P(x), \mu_Q(y)))$, it is

$$(\mu_P \rightarrow \mu_Q)(x_i, y_j) = J(\mu_P(x_i), \mu_Q(y_j)) = J(r_i, s_j) = a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m.$$  

With all that,

$$\mu_{Q^*}(y_j) = \sup_{1 \leq i \leq n} T_0(\mu_{P^*}(x_i), J(\mu_P(x_i), \mu_Q(y_j))) = \max_{1 \leq i \leq n} T_0(r_i^*, J(r_i, s_j)) =$$

$$= \max_{1 \leq i \leq n} T_0(r_i^*, a_{ij}), 1 \leq i \leq n,$$

since in the finite case the sup is just max.

Calling $\mu_{Q^*} = s_1^*/y_1 + \ldots + s_m^*/y_m$, it results

$$s_j^* = \max_{1 \leq i \leq n} T_0(r_i^*, a_{ij}), 1 \leq j \leq m,$$

showing an special ‘composition’ of the matrices

$$(r_1^*, \ldots, r_n^*) = [\mu_{P^*}] \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix} = [J],$$

in which the elements of the classical product of matrices (rows by columns) $\sum_{1 \leq i \leq n} r_i^*.a_{ij}$ are substituted by $\max_{1 \leq i \leq n} T_0(r_i^*, a_{ij})$.

This composition is called the max-$T_0$ product of matrices, instead of the classical sum-prod composition. Hence,

$$[\mu_{Q^*}] = (s_1^*, \ldots, s_m^*) = [\mu_{P^*}] \otimes [J],$$

gives the CRI’s output.
3.4. INFERENCE WITH FUZZY RULES

Example 3.4.5. With $\mu_P = 0.7/x_1 + 0.8/x_2 + 1/x_3$, $\mu_Q = 0.9/y_1 + 0.6/y_2 + 0.8/y_4$, $\mu_{P^*} = 0.6/x_1 + 0.7/x_2 + 1/x_3$, and $J(a, b) = \min(1, 1 - a + b)$, follows:

$a_{11} = J(0.7, 0.9) = \min(1, 1 - 0.7 + 0.9) = 1$; $a_{12} = J(0.7, 0.6) = 0.9$; $a_{13} = J(0.7, 0.0) = 0.3$; $a_{14} = J(0.7, 0.8) = 1$; $a_{21} = J(0.8, 0.9) = 1$; $a_{22} = J(0.8, 0.6) = 0.8$; $a_{23} = J(0.8, 0) = 0.2$; $a_{24} = J(0.8, 0.8) = 1$; $a_{31} = J(1, 0.9) = 0.9$; $a_{32} = J(1, 0.6) = 0.6$; $a_{33} = J(1, 0) = 0$; $a_{34} = J(1, 0.8) = 0.8$.

Hence,

$$[\mu_{Q^*}] = (0.6 0.7 1) \otimes \begin{pmatrix} 1 & 0.9 & 0.3 & 1 \\ 1 & 0.8 & 0.2 & 1 \\ 0.9 & 0.6 & 0 & 0.8 \end{pmatrix} = (0.9 \ 1 \ 0.9 \ 0.8),$$

since: $\max(W(0.6, 1), W(0.7, 1), W(1, 0.9)), \max(W(0.6, 0.9), W(0.7, 0.8), W(1, 0.6)), \max(W(0.6, 0.3), W(0.7, 0.2), W(1, 0)), \max(W(0.6, 1), W(0.7, 1), W(1, 0.8)) = (\max(0.6, 0.7, 0.9), \max(1, 0.6), \max(0.9, 0.9, 0), \max(0.6, 0.7, 0.8)) = (0.9 \ 1 \ 0.9 \ 0.8)$. That is

$$\mu_{Q^*} = 0.9/y_1 + 1/y_2 + 0.9/y_3 + 0.8/y_4.$$ 

In the case $\mu_P$ is interpreted $P = more or less big$, $\mu_Q$ is interpreted $Q = not very big$, and $\mu_{P^*}$ is interpreted $P^* = medium$, it is possible to agree on $Q^* = more or less big$.

3.4.2

Actually, there are no systems described by a single rule. What to do when a system is described by, at least, two rules? With, for example

- r1: If $x$ is $P_1$, then $y$ is $Q_1$
- r2: If $x$ is $P_2$, then $y$ is $Q_2$,

an input $\mu_{P^*}$ gives
• with r1, the output $\mu_{Q_1^*}$, obtained by using CRI
• with r2, the output $\mu_{Q_2^*}$, obtained by using CRI.

Provided the firing of the rules corresponds to ‘fire r1 or fire r2’, then the final output is given by

$$\mu_{Q^*} = \max(\mu_{Q_1^*}, \mu_{Q_2^*})$$

and analogously for more than two rules. For example, in the case of p rules $r_1, \ldots, r_p$, the result will be

$$\mu_{Q^*} = \max(\mu_{Q_1^*}, \ldots, \mu_{Q_p^*})$$

where $\mu_{Q_i^*} (1 \leq i \leq p)$ is the output obtained with the rule $r_i$ and the input $\mu_P$.

Example 3.4.6. With $X = [0, 1], Y = [0, 10]$, consider the rules

- If $x$ is big, then $y = 2$
- If $x$ is small, then $y = 8$
- If $x$ is around 0.5, then $y = 6$,

and the input $x_0 = 0.4$. Which is the final output of this system if the rules are represented by $J(a, b) = a \cdot b$?

$$\mu_{Q_1^*}(y) = J(\mu_B(0.4), \mu_{\{2\}}(y)) = 0.4 \cdot \mu_{\{2\}}(y) = \begin{cases} 0.4, & \text{if } y = 2 \\ 0, & \text{if } y \neq 2 \end{cases}$$

$$\mu_{Q_2^*}(y) = J(\mu_5(0.4), \mu_{\{8\}}(y)) = 0.6 \cdot \mu_{\{2\}}(y) = \begin{cases} 0.6, & \text{if } y = 8 \\ 0, & \text{if } y \neq 8 \end{cases}$$

$$\mu_{Q_3^*}(y) = J(\mu_{A0.5}(0.4), \mu_{\{6\}}(y)) = \mu_{A0.5}(0.4) \cdot \mu_{\{6\}}(y) = \begin{cases} \mu_{A0.5}(0.4), & \text{if } y = 6 \\ 0, & \text{if } y \neq 6, \end{cases}$$
3.4. INFERENCE WITH FUZZY RULES

with \( \mu_B(x) = x \), and \( \mu_S(x) = 1 - x \). Taking as \( \mu_{A0.5} \) the triangular function

![Diagram](image)

and since the left side equation is \( y = \frac{x - 0.35}{0.15} \), it is \( \mu_{A0.5}(0.4) = 0.3 \). Hence,

\[
\mu_{Q^*}(y) = \begin{cases} 
0.3, & \text{if } y = 6 \\
0, & \text{if } y \neq 6,
\end{cases}
\]

Finally

\[
\mu_{Q^*}(y) = \max(\mu_{Q_1^*}(y), \mu_{Q_2^*}(y), \mu_{Q_3^*}(y)) = \begin{cases} 
0.4, & \text{if } y = 2 \\
0.3, & \text{if } y = 6 \\
0.6, & \text{if } y = 8 \\
0, & \text{otherwise}
\end{cases}
\]

graphically,

![Diagram](image)

In many applications, the output should be converted into a single numerical value: it should be ‘defuzzified’. In this cases with numerical input and
numerical rule’s consequents (the most used in fuzzy control), such number is easily obtained by averaging the values of $\mu_{Q^*}$, in the form

$$\frac{2 \times 0.4 + 6 \times 0.3 + 8 \times 0.6}{0.4 + 0.3 + 0.6} = \frac{7.5998}{1.3333} = 5.6999 \approx 5.7$$

Hence, the numerical output that corresponds to the input $x_0 = 0.4$, is $y_0 = 5.7$.

**Remark 3.4.7.** Notice that once a system of rules linguistically describing the behavior of a system is given, and where the consequents of the rules are numerical, at each numerical input $x_0$ in $X$ does correspond a numerical output $y_0$ in $Y$. In that way, a function CRI: $X \rightarrow Y$ is defined. As it will be later on commented, were the system’s behaviour previously known by a continuous function $f : X \rightarrow Y$, the function CRI approaches, under some additional conditions, the function $f$.

**Remark 3.4.8.** Look how important is to properly select the T-conditionals representing the rules.

Given the rule ‘If $x$ is small, then $y$ is big’, with $X = Y = [0, 1]$, and $\mu_S(x) = 1 - x, \mu_B(y) = y$, $J(a, b) = \max(1 - a, b)$, it follows $J(\mu_S(x), \mu_B(y)) = \max(x, y)$, that could be interpreted as ‘$x$ is high or $y$ is big’.

With $J(a, b) = \min(1, 1 - a + b)$, it follows $J(\mu_S(x), \mu_B(y)) = \min(1, x + y) = W^*(x, y)$, also interpretable as ‘$x$ is big or $y$ is big’. With $J(a, b) = \min(a, b)$, is $J(\mu_S(x), \mu_B(y)) = \min(1 - x, y)$, interpreted as ‘$x$ is small and $y$ is big’

### 3.4.3

Let’s consider more examples.

**Example 3.4.9.** Rule ‘If $x$ is big, then $y = 0.8$’, with $x, y$ in $[0, 1]$, and the observation $x \in [0.4, 0.6]$. Hence:
### 3.4. INFERENCE WITH FUZZY RULES

\[ J(\mu_B(x), \mu_{[0.8]}(y)) = x\mu_{[0.8]}(y) = \begin{cases} x, & \text{if } y = 0.8 \\ 0, & \text{if } y \neq 0.8 \end{cases} \]

Then, \( \mu_{Q^*}(y) = \sup_{x \in [0,1]} \min(\mu_{[0.4,0.6]}(x), x\mu_{[0.8]}(y)) = \begin{cases} 0.6, & \text{if } y = 0.8 \\ 0, & \text{if } y \neq 0.8 \end{cases} \)

since,

**Example 3.4.10.** Rule: ‘If \( x \) is big, then \( y \) is small’, with the same observation as that in last example and with \( \mu_B(x) = x \), \( \mu_S(y) = 1 - y \), and \( J(a, b) = \min(a, b) \), follows:

\( \mu_{Q^*}(y) = \sup_{x \in [0,1]} \min(\mu_{[0.4,0.6]}(x), \min(x, 1-y)) = \sup_{x \in [0,1]} \min(\min(\mu_{[0.4,0.6]}(x), x), 1-y) = \min(x, 1-y) = \min(0.6, 1-y). \)

**Example 3.4.11.** \( X = \{1, 2, 3\}, Y = \{6, 7\} \). Rule: ‘If \( x \) is around 2, then \( y = 6 \)’, and \( \mu_{P^*}(x) = 0.6/1 + 0.9/2 + 0.7/3 \), with \( J(a, b) = ab \) (Larsen). It results

\( (\mu_{Q^*}(6), \mu_{Q^*}(7)) = (0.6 \ 0.9 \ 0.7) \otimes \begin{pmatrix} 0.5 & 0 \\ 1 & 0 \\ 0.5 & 0 \end{pmatrix} = (0.9 \ 0.7) \), that is \( \mu_{Q^*} = 0.9/6 + 0/7. \)

**Example 3.4.12.** With \( x \in [0,1] \), and \( y \in [0,1] \), consider

- r1: ‘If \( x \) is big, then \( y \) is small’, represented by \( J_1(a, b) = ab \)
• r2: 'If $x$ is very small, then $y$ is very big', represented by $J_2(a, b) = \min(a, b)$

Consider $x_0 = 0.4$, $\mu_B(x) = x$, $\mu_S(y) = 1-y$, $\mu_{VS}(x) = (1-x)^2$, $\mu_{VB}(y) = y^2$. Then

• $J_1(\mu_B(x), \mu_S(y)) = x(1-y)$
• $J_2(\mu_{VS}(x), \mu_{VB}(y)) = \min((1-x)^2, y^2) = [\min(1-x, y)]^2$.

Hence,

• $\mu_{Q_1^*}(x) = J_1(0.4, 1-y) = 0.4(1-y)$,

• $\mu_{Q_2^*}(x) = J_2((1-0.4)^2, y^2) = (\min(0.6, y))^2 = \begin{cases} 0.36, & \text{if } y \geq 0.6 \\ y^2, & \text{if } y < 0.6, \end{cases}$

whose graphics are,

Finally,

$$\mu_{Q^*}(y) = \max(\mu_{Q_1^*}(y), \mu_{Q_2^*}(y)) = \begin{cases} 0.4(1-y), & \text{if } 0 \leq y \leq 0.463 \\ y^2, & \text{if } 0.463 \leq y \leq 0.6 \\ 0.36, & \text{if } 0.6 \leq y \leq 1. \end{cases}$$

**Example 3.4.13.** Let’s find the function CRI: $X \rightarrow Y$, in the case with $X = [0, 1]$, $Y = [0, 1]$, and

• r1: If $x$ is small, then $y = 9$
• r2: If $x$ is big, then $y = 2$,

it follows $\mu_{Q_1^*}(y) = (1-x)\mu_{(9)}(y) = \begin{cases} 1-x, & \text{if } y = 9 \\ 0, & \text{if } y \neq 9, \end{cases}$

$\mu_{Q_2^*}(y) = x\mu_{(2)}(y) = \begin{cases} x, & \text{if } y = 2 \\ 0, & \text{if } y \neq 2, \end{cases}$

and

$\mu_{Q^*}(y) = \max(\mu_{Q_1^*}(y), \mu_{Q_2^*}(y)) = \begin{cases} x, & \text{if } y = 2 \\ 1-x, & \text{if } y = 9 \\ 0, & \text{otherwise} \end{cases}$

that gives,

$$CRI(x) = \frac{2x + 9(1-x)}{x + 1-x} = 9 - 7x,$$

as the “theoretical” (linear) behavior of the system $(x, y)$. For each $x_0 \in X$, the value $CRI(x_0) \in Y$ is the defuzzified value that corresponds to $x_0$.

Remark 3.4.14. Systems of fuzzy rules behave as universal approximations. This means the following. Suppose a system $(x, y)$, with $x \in [a, b]$, $y \in [c, d]$, that behave by following the continuous function $f(x) = y$. For each $\varepsilon > 0$, there is always a system of fuzzy rules and a defuzzification method for the output, giving a function $\text{CRI}: X \to Y$ such that

$$|f(x) - CRI(x)| < \varepsilon, \text{ for all } x \in [a, b]$$
This theorem is simply an existential one, since there is no general method for obtaining neither a fuzzy representation of the system of rules, not the defuzzification method. It simply shows that CRI approaches enough well $f$ for all points in $[a, b]$.

### 3.5

How to defuzzify non discrete outputs $\mu_{Q^*}$? Let us proceed with two examples without computational difficulties.

**1st Example.** Rules,

- r1: If $x$ is big, then $y$ is small
- r2: If $x$ is small, then $y$ is big

with $X = [0, 1]$, and $Y = [0, 10]$. Take,

$$\mu_B(x) = x, \quad \mu_S(y) = 1 - \frac{y}{10}, \quad \mu_S(x) = 1 - x, \quad \mu_B(y) = \frac{y}{10}. $$

and $J(a, b) = \min(a, b)$ -Mamdani-. Notice that, with the observation $x_0 = 0.5$,

$$\mu_{Q_1^*}(y) = \min(0.5, 1 - \frac{y}{10}), \quad \mu_{Q_2^*}(y) = \min(1 - 0.5, \frac{y}{10}).$$

Then, $\mu_{Q^*}(y) = \max(\min(0.5, 1 - \frac{y}{10}), \min(0.5, \frac{y}{10})) = \max(0.5, \min(1 - \frac{y}{10}, \frac{y}{10})) = 0.5$, since $\min(1 - \frac{y}{10}, \frac{y}{10}) \leq 0.5$.

The area below $\mu_{Q^*}(y) = 0.5$, is $A = 0.5 \times 10 = 5$ square units. Hence, a way to defuzzify $\mu_{Q^*}$ consists of searching the center of area, that is, a point $y_0 \in [0, 10]$ such that

$$\int_0^{y_0} 0.5dy = \frac{A}{2} = 2.5, \text{ or } \int_0^{y_0} dy = 5.$$

Hence, $[y]_{0}^{y_0} = y_0 = 5$. The defuzzified value corresponding to $x_0 = 0.5$, is $y_0 = 5$. The method, when the conditional is Mamdani, is graphically reflected as follows.
2nd Example. Identical to the first example, but with the input \( x_0 = 0.3 \).

It is

\[
\mu_{Q_1^*}(y) = \min(0.3, 1 - \frac{y}{10}), \quad \mu_{Q_2^*}(y) = \min(1 - 0.3, \frac{y}{10}).
\]

Hence,

\[
\mu_{Q^*}(y) = \begin{cases} 
0.3, & \text{if } 0 \leq y \leq 3 \\
\frac{y}{10}, & \text{if } 3 \leq y \leq 7 \\
0.7, & \text{if } 7 \leq y \leq 10
\end{cases}
\]

that is graphically find as follows:
The area below $\mu_{Q^*}$ is $A = \text{rectangle}(1) + \text{rectangle}(2) + \text{triangle}(3) = 0.3 \times 7 + 3 \times 0.7 + \frac{4 \times 0.4}{2} = 5$. Since the area of the rectangle with base $[0, 3]$ is $0.3 \times 3 = 0.9$, it is $y_0 > 3$. Hence,

$$\int_0^3 0.3\,dy + \int_{3}^{y_0} \frac{y}{10}\,dy = A/2 = 2.5.$$ \[1\]

Then $3 \times 0.3 + \frac{1}{10} \int_{3}^{y_0} y\,dy = 2.5$, or $\int_{3}^{y_0} y\,dy = 10(2.5 - 0.9) = 16$. Thus,

$$\left[ \frac{y^2}{2} \right]_3^{y_0} = 16 \Rightarrow y_0^2 - 9 = 32 \Rightarrow y_0^2 = 41 \Rightarrow y_0 = 6.4.$$ \[2\]

The defuzzified value that corresponds to $x_0 = 0.3$, is $y_0 = 6.4$.

**Comments to Examples 1 and 2.** Defuzzifying with the centre of area we obtained an output for all values $x_0$ in $[0, 1]$. Let’s see it by means of the function CRI with defuzzification made by the centre of area.

1. The graphics, for any input $x_0 \leq 1/2$, is
The area below $\mu_{Q^*}$ is

$$A = 10(1-x_0)x_0 + (10 - 10(1-x_0))(1-x_0) + \frac{(1 - x_0 - x_0)(10(1 - x_0) - 10x_0)}{2}$$

then,

$$2A = 20(1-x_0)x_0 + 20(1-x_0)x_0 + (1-2x_0)(10-20x_0),$$

and

$$A = 20x_0(1-x_0) + 5 - 20x_0 + 20x_0^2 = 5.$$  

Hence,

$$10x_0^2 + \int_{10x_0}^{y_0} \frac{y}{10} \, dy = 2.5 \Rightarrow y_0 = \sqrt{50 - 100x_0^2},$$

and $CRI(x) = \sqrt{50 - 100x^2},$ if $x \leq 1/2$. For example, $CRI(0.5) = 0.5$, and $CRI(0.3) = \sqrt{41} = 6.4$, as it was shown. It is also $CRI(0.1) = 7$.

Last formula, $CRI(x) = \sqrt{20 - 100x^2},$ gives real values provided $20 - 100x^2 \geq 0,$ or $x^2 \leq 1/5$. Since, it is $x \leq 1/5$, that implies $x^2 \leq 1/4 \leq 1/2$, and it follows that the formula is useful for all $x \in [0, 1]$ such that $x \leq 1/2$.

2. For any input $x_0 \geq 1/2$, the graphic is
and the area below $\mu_{Q^*}$ is

\[
A = 10x_0(1 - x_0) + (10 - 10x_0)x_0 + \frac{(10x_0 - 10(1-x_0))(x_0 - 1 + x_0)}{2} = 20x_0(1 - x_0) + (10x_0 - 5)(2x_0 - 1) = 5.
\]

Hence, $A/2 = 2.5$, and

\[
10(1-x_0)^2 + \int_{10(1-x_0)}^{y_0} \frac{y}{10} dy = 2.5, \text{ or } 100(1-x_0)^2 + \int_{10(1-x_0)}^{y_0} y dy = 2.5,
\]

giving $y_0^2 = 50 - 100(1 - x_0)$, or $y_0 = \sqrt{50 - 100(1 - x_0)^2}$, that gives real values provided $50 - 100(1 - x_0)^2 \geq 0$, equivalent to $1/2 \geq (1 - x_0)^2$, or to $x_0 \geq 1 - \sqrt{1/2}$. Since, $1/2 \leq x$, it is $(1 - x)^2 \leq 1/4 \leq 1/2$. Then,

$$CRI(x) = \sqrt{50 - 100(1 - x)^2}, \text{ if } 1/2 \leq x.$$ 

For example,

$$CRI(0.7) = \sqrt{41} = 6.4, \quad CRI(0.8) = \sqrt{46} = 6.78, \quad CRI(0.9) = \sqrt{49} = 7.$$ 

3. Finally, with defuzzification by the centre of area, is:

$$CRI(x) = \begin{cases} 
\sqrt{50(1 - 2x^2)}, & \text{if } x \leq 1/2 \\
\sqrt{50(1 - 2(1 - x)^2)}, & \text{if } x \geq 1/2. 
\end{cases}$$

Notice that $CRI(0) = \sqrt{50}$, $CRI(0.5) = \sqrt{50(1 - 20.5^2)} = \sqrt{25} = 5$, and $CRI(1) = \sqrt{50}$.

The graphic of CRI is
3.6

As it was said before, the output is a logical consequence of the premises given by the input and the rule. Notwithstanding, the situation is different if, taking the rule as defining the system, only the input is considered as a premise. But, before to consider this question, let us consider what happens when there is more than a single rule, a situation that, as it was also said is not a realistic one.

1. If $\mu_{Q_1^*}, \mu_{Q_2^*} \in Cons(\{\mu_{P*}\})$, from $\mu_{P*} \leq \max(\mu_{Q_1^*}, \mu_{Q_2^*}) = \mu_{Q^*}$, follows $\mu_{Q^*} \in Cons(\{\mu_{P*}\})$.

2. If $\mu_{Q_1^*}$ or $\mu_{Q_2^*}$ is a conjecture of $\{\mu_{P*}\}$, then $\mu_{Q^*} \in Conj(\{\mu_{P*}\})$.

The proof follows in this way, provided it is, for instance, $\mu_{Q_1^*} \in Conj(\{\mu_{P*}\})$,

- $\mu_{Q^*} = \max(\mu_{Q_1^*}, \mu_{Q_2^*}) \Rightarrow \mu_{Q_1^*} \leq \mu_{Q^*}, \mu_{Q_2^*} \leq \mu_{Q^*} \Rightarrow \mu'_{Q_1^*} \leq \mu'_{Q_1^*}, \mu'_{Q_2^*} \leq \mu'_{Q_2^*} \Rightarrow \mu'_{Q^*} \leq \min(\mu'_{Q_1^*}, \mu'_{Q_2^*})$.
- If $\mu_{P*} \leq \mu'_{Q^*}$, then $\mu_{P*} \leq \mu'_{Q_1^*}$, that is absurd. Hence, it is $\mu_{P*} \not\leq \mu'_{Q^*}$, and $\mu_{Q^*} \in Conj(\{\mu_{P*}\})$.

In conclusion

- If all the partial outputs $\mu_{Q_1^*}, \mu_{Q_2^*}, \ldots, \mu_{Q_n^*}$, are consequences of the input $\mu_{P*}$, also the final output $\mu_{Q^*}$ is a consequence of $\mu_{P*}$.
- If at least one of the partial outputs $\mu_{Q_i^*}$ $(1 \leq i \leq n)$ is just a conjecture of the input $\mu_{P*}$, also the final output $\mu_{Q^*}$ is a conjecture of $\mu_{P*}$.

Nevertheless, it is not usual that $\mu_{Q^*}$ results to be a consequence of $\mu_{P*}$.

Let’s us introduce a necessary and sufficient condition for it in the particular case in which there is only one rule represented by $J(a, b) = ab$ (Larsen).
Let it be “If $x$ is $P$, then $y$ is $Q$” ($x, y \in X$), and $(\mu_P \rightarrow \mu_Q)(x, y) = \mu_P(x)\mu_Q(y)$, with the input $x = x_0$. Then,

$$\mu_{Q^*}(y) = \mu_{P^*}(x_0)\mu_Q(y), \quad \forall y \in X.$$ 

Provided $\mu_{\{x_0\}} \neq \mu_0$, to have $\mu_{\{x_0\}}(y) \leq \mu_{Q^*}(y) = \mu_{P^*}(x_0)\mu_Q(y)$, it is necessary that, with $y = x_0$, $1 \leq \mu_{P^*}(x_0)\mu_{Q^*}(x_0)$ or $1 = \mu_{P^*}(x_0) = \mu_{Q^*}(x_0)$. Hence, $\mu_{Q^*} \in Cons(\{\mu_{\{x_0\}}\})$ implies $\mu_{P^*}(x_0) = \mu_{Q^*}(x_0) = 1$.

Provided $\mu_{P^*}(x_0) = \mu_{Q^*}(x_0) = 1$, from $\mu_{Q^*}(y) = \mu_{P^*}(x_0)\mu_Q(y)$, follows $\mu_{Q^*}(y) = \mu_Q(y)$, for all $y \in X$, and,

- If $y = x_0$, $\mu_{\{x_0\}}(x_0) = 1 = \mu_{Q^*}(x_0)$
- If $y \neq x_0$, $\mu_{\{x_0\}}(y) = 0 \leq \mu_{Q^*}(y)$,

that is

$$\mu_{\{x_0\}}(y) \leq \mu_{Q^*}(y), \quad \forall y \in X.$$ 

Hence, in this particular case, the necessary and sufficient condition for being $\mu_{Q^*} \in Cons(\{\mu_{\{x_0\}}\})$ is that $\mu_{P^*}(x_0) = \mu_{Q^*}(x_0) = 1$. Nevertheless, what happens in most of the cases is that $\mu_{Q^*} \in Conj(\{\mu_{\{x_0\}}\})$, with $\mu_{Q^*} \in Sp(\{\mu_{\{x_0\}}\})$, or $\mu_{Q^*} \in Hyp(\{\mu_{\{x_0\}}\})$. Let’s show an example in which the output is a speculation of the input and other in which the output is a hypothesis.

**Example.** Rule, “If $x$ is small, then $y$ is big”, with $X = Y = [0, 10]$ and $J(a, b) = ab$ (Larsen), with $\mu_S(x) = 1 - \frac{x}{10}$, $\mu_B(y) = \frac{y}{10}$, and $x_0 = 5$. It is

$$\mu_{Q^*}(y) = J\left(1 - \frac{5}{10}, \frac{y}{10}\right) = \frac{y}{20},$$

and from the graphic
it is clear that $\mu_{Q^*}$ is not comparable with $\mu_{[5]}$ ($\mu_{Q^*}$ NC $\mu_{[5]}$), and that $\mu_{[5]} \preceq \mu_{Q^*} = 1 - \mu_{Q^*}$. Hence,

$$\mu_{Q^*} \in Conj(\{\mu_{[5]}\}),$$
and namely $\mu_{Q^*} \in Sp(\{\mu_{[5]}\})$.

Notice that $\mu_{Q^*}(5) = \frac{1}{4} \neq 1$.

**Example.** Rule, “If $x$ is big, then $y$ is very big”, with $X = Y = [0, 10]$ and the observation “$x$ is constantly equal to 0.8” for all $x \in [0, 1]$.

Taking

$$\mu_B(x) = x, \quad \mu_{vB}(y) = y^2, \quad J(a,b) = \min(a,b) \text{ (Mamdani)},$$

follows

$$\mu_{Q^*}(y) = \min(\mu_{P^*}(0.8), \mu_{vB}(y)) = \min(0.8, y^2).$$

Graphically,
since $y^2 = 0.8$ means $y = \sqrt{0.8}$. Hence, it is $\mu_0 \neq \mu_{Q^*} \leq \mu_{P^*}$, and $\mu_{P^*} = \mu'_{Q^*} = 1 - \mu_{Q^*}$, that imply $\mu_{Q^*} \in \text{Conj}(\{\mu_{P^*}\})$, and namely $\mu_{Q^*} \in \text{Hyp}(\{\mu_{P^*}\})$.

Notice that this second example contains the actually rare observation that the input is constant.

**Last remark.**

For any continuous t-norm $T$, the function $\mu : Y \to [0, 1]$, defined by

$$\mu(y) = \sup_{x \in X} T(\mu_{P^*}(x), J(\mu_P(x), \mu_Q(y))), \forall y \in Y,$$

does verify

$$T(\mu_{P^*}(x), J(\mu_P(x), \mu_Q(y))) \leq \mu(y), \forall y \in Y, x \in X,$$

that is, $\mu \in \text{Cons}(\{\mu_{P^*}, \mu_P \to \mu_Q\})$. Nevertheless, if $T = T_0$, the continuous t-norm for which $J$ is a $T_0$-conditional, that is, such that

$$T_0(\mu_P(x), J(\mu_P(x), \mu_Q(y))) \leq \mu(y), \forall y \in Y, x \in X,$$

it could be that when $\mu_{P^*} = \mu_P$, then $\mu \neq \mu_Q$. A undesiderable situation, because fuzzy logic must contain all classical cases.

For example, with the rule ‘If $x$ is small, then $y$ is big’ ($X = Y = [0, 1]$), and $J(a, b) = \max(1 - a, b)$ that is a $W$-conditional, taking $\mu_S(x) = 1 - x$ and $\mu_B(y) = y$, follows:

- With $T = W$, $\mu(y) = \sup_{x \in [0, 1]} W(1 - x, \max(x, y)) = \sup_{x \in [0, 1]} (0, y - x) = y = \mu_B(y)$.
- With $T = \text{prod}$, $\mu(y) = \sup_{x \in [0, 1]} (1 - x) \max(x, y) = \sup_{x \in [0, 1]} \max((1 - x)x, (1 - x)y) = y = \max(1/4, y/2)$, not coincidental with $\mu_B$.
- with $T = \text{min}$, $\mu(y) = \sup_{x \in [0, 1]} \min((1 - x), \max(x, y)) = 1$, or $\mu = \mu_1$, also not coincidental with $\mu_B$. 
Hence, although with any continuous t-norm $T$, an output $\mu$ is obtained, if this $T$ does not make $J$ a $T$-conditional it is not sure that $P = P^*$ implies $Q = Q^*$. It is necessary to take $T_0$ in the CRI!
Chapter 4

Fuzzy relations

4.1 What is a fuzzy relation?

A predicate $R$ in a cartesian product $X_1 \times \ldots \times X_n$ is called a relational (n-array) predicate. For example, if $X_1 = X_2 = [0, 10]$, $R=$close to, $(x, y)$ is $R$ or ‘$x$ is close to $y’$, is a relational binary predicate.

Analogously, if $X_1 = X_2 = London$, R=lives in the same borough, or ‘$x$ lives in the same borough than $y’$, is a relational binary predicate.

A fuzzy relation in $X_1 \times \ldots \times X_n$ is any function $\mu : X_1 \times \ldots \times X_n \to [0, 1]$. If interpreting $\mu_R(x_1, \ldots, x_n) =$‘degree up to which $(x_1, \ldots, x_n)$ is in $R’$, it is said that $\mu_R$ represents the n-array relational relation $R’$.

Any rule ‘If $x$ is $P$, then $y$ is $Q’ defines the binary predicate $Q/P$ in $X \times Y$ given by

$$(x, y) \text{ is } Q/P \iff \text{If } x \text{ is } P, \text{ the } y \text{ is } Q,$$

whose representation, or membership function of the corresponding fuzzy sets $Q/P$ is given by

$$\mu_{Q/P}(x, y) = (\mu_P \to \mu_Q)(x, y) = J(\mu_P(x), \mu_Q(y)),$$

once a T-conditional $J$ adapted to the meaning of $Q/P$ is selected. A fuzzy relation $\mu_R$ is nothing else than a fuzzy set in $X_1 \times \ldots \times X_n$. 

163
For example, if $R = \text{closer to}$, is represented by

$$\mu_R(x, y) = \max(0, k|x - y|), \text{ for all } x, y \in [0, 1],$$

with $k \in (0, 1)$ a parameter chosen at each particular case, it is $\mu_R(0, 0) = 0$, $\mu_R(1, 1) = 0$, $\mu_R(0, 1) = \mu_R(1, 0) = \max(0, k) = k$, $\mu_R(1/2, 1) = \mu_R(1, 1/2) = \max(0, k/2) = k/2$, etc, with the graphic,

When the sets $X_1, \ldots, X_n$ are finite, as it is in the case $n = 2$, $\mu_R$ is reduced to a matrix. For example if $X_1 = \{x_1, \ldots, x_p\}$, and $X_2 = \{y_1, \ldots, y_q\}$, then

$$\mu_R(x_i, y_j) = r_{ij}, \ 1 \leq i \leq n, 1 \leq j \leq m, \text{ or, } \mu_R = r_{11}/(x_1, y_1) + \ldots + r_{nm}/(x_n, y_m),$$

that gives the $n \times m$ matrix

$$[R] = \begin{pmatrix}
r_{11} & r_{12} & \cdots & r_{1m} \\
r_{21} & r_{22} & \cdots & r_{2m} \\
& \vdots & \ddots & \vdots \\
r_{n1} & r_{n2} & \cdots & r_{nm}
\end{pmatrix}.$$
4.2. HOW TO COMPOSE FUZZY RELATIONS?

In the finite case there is again another representation of a fuzzy relation by means of a directed graph. For example, if \( X_1 = \{x_1, x_2\} \) and \( X_2 = \{y_1, y_2, y_3\} \), the fuzzy relation \( R = \begin{pmatrix} 0.5 & 0.7 & 1 \\ 0.8 & 0 & 0.8 \end{pmatrix} \), corresponds to the directed graph

\[ \begin{array}{c}
X_1 \\
\downarrow 0.5 \\
\downarrow 0.8 \\
X_2 \\
\downarrow 0.7 \\
\downarrow 1 \\
X_3 \\
\downarrow 0.8 \\
Y_1 \\
\downarrow \\
Y_2 \\
\downarrow \\
Y_3 \\
\end{array} \]

4.2 How to compose fuzzy relations?

Given two fuzzy relations \( \mu : X \times Q \rightarrow [0, 1] \), and \( \sigma : Y \times Z \rightarrow [0, 1] \), how can we obtain a relation \( \lambda : X \times Z \rightarrow [0, 1] \) through \( \mu \) and \( \sigma \)? To solve this problem, there is the \( \text{Sup}_T \) product of fuzzy relations, given by

\[ \lambda(x, z) = \text{Sup}_{y \in X} T(\mu(x, y), \sigma(y, z)), \text{ for all } (x, z) \in X \times Z, \]

a formula that, in the finite case \( X = \{x_1, ..., x_n\} \), \( Y = \{y_1, ..., y_m\} \), \( Z = \{z_1, ..., z_p\} \), reduces to,
\[ \lambda(x_i, z_j) = \max_{1 \leq k \leq m} T(\mu(x_i, y_k), \sigma(y_k, z_j)). \]

Provided \([\mu] = (r_{ik}), [\sigma] = (s_{kj}),\) then

\[ t_{ij} = \lambda(x_i, z_j) = \max_{1 \leq k \leq m} T(r_{ik}, s_{kj}), \quad 1 \leq i \leq n, \quad 1 \leq j \leq p, \]

giving the matrix \([\lambda] = (t_{ij})\) as the Max-T product, or composition, of the matrices \([r_{ik}]\) and \([s_{kj}]\), that was above introduced: \((t_{ij}) = (r_{ik}) \otimes_T (s_{kj})\).

**Example 4.2.1.** \(^1\)Let

- \(X = \{p_1, \ldots, p_4\}\), a set of patients
- \(Y = \{s_1, s_2, s_3\}\), a set of symptoms
- \(Z = \{d_1, \ldots, d_5\}\), a set of deceases,

and the fuzzy relation \(\sigma\)

\[
[\sigma] = \begin{pmatrix}
0.7 & 0 & 0 & 0.3 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 & 0 \\
0 & 0.7 & 0.2 & 0.9 & 0
\end{pmatrix}
\]

showing the medical knowledge of how strongly each symptom is associated with a decease. Suppose also that, by examining the patients, the doctors conclude the matrix

\[
[\mu] = \begin{pmatrix}
0 & 0.3 & 0.4 \\
0.2 & 0.5 & 0.3 \\
0.8 & 0 & 0 \\
0.7 & 0.7 & 0.9
\end{pmatrix}
\]

that describes numerically how strongly the symptoms are manifested in the patients. Then,

\[
[\lambda] = [\mu] \otimes_{\min} [\sigma]
\]

is the matrix expressing the association patients/deceases, and facilitates a medical diagnose. That is,

\[
[\lambda] = \begin{pmatrix}
0 & 0.3 & 0.4 \\
0.2 & 0.5 & 0.3 \\
0.8 & 0 & 0 \\
0.7 & 0.7 & 0.9
\end{pmatrix} \otimes_{\min} \begin{pmatrix}
0.7 & 0 & 0.3 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0 & 0.7 & 0.2 & 0.9 \\
0 & 0.7 & 0.7 & 0.9
\end{pmatrix} = \begin{pmatrix}
0.3 & 0.4 & 0.3 & 0.4 & 0 \\
0.5 & 0.5 & 0.5 & 0.4 & 0.2 \\
0.7 & 0 & 0 & 0.3 & 0.6 \\
0.7 & 0.7 & 0.7 & 0.9 & 0.6
\end{pmatrix},
\]

where, f.ex.,

\[
t_{43} = \max(\min(0.7, 0), \min(0.7, 0.8), \min(0.9, 0.2)) = \max(0, 0.7, 0.2) = 0.7.
\]

The matrix \([\lambda]\) results from a mixing between knowledge and observation.

**Remarks 4.2.2.** 1. As it is easy to prove, it is

\[
([\mu] \otimes_T [\sigma])^t = [\sigma]^t \otimes_T [\mu]^t,
\]

with the matrices \([\sigma]^t, [\mu]^t\), defined by \(\sigma^t(x, y) = \sigma(y, x), \mu^t(x, y) = \mu(y, x)\), giving

\[
([\mu]^t)^t = [\mu].
\]

The matrix \([\mu]^t\) is the transposed of \([\mu]\).

2. In general, the max-T composition is associative, but not commutative. That is, *if the compositions* \([\mu] \otimes_T ([\sigma] \otimes_T [\lambda])\), and \(([\mu] \otimes_T [\sigma] \otimes_T [\lambda])\), *are possible* it is,

\[
([\mu] \otimes_T [\sigma]) \otimes_T [\lambda] = [\mu] \otimes_T ([\sigma] \otimes_T [\lambda]),
\]

but, in general, \([\mu] \otimes_T [\sigma] \neq [\sigma] \otimes_T [\mu]\).

### 4.3 Which relevant properties do have a fuzzy binary relation?

The most relevant properties of a fuzzy relation \(\mu : X \times X \to [0, 1]\) are the following,
1. Reflexive property, $\mu(x, x) = 1$, for all $x \in X$.

2. Symmetric property, $\mu(x, y) = \mu(y, x)$, for all $x, y \in X$, implies $x = y$.

3. Antisymmetric property, $\mu(x, y) > 0, \mu(y, x) > 0$ implies $x = y$.

4. T-transitive property $T(\mu(x, y), \mu(y, z)) \leq \mu(x, z)$, for all $x, y, z \in X$, and some continuous t-norm $T$.

In the finite case, for what concerns properties reflexive and symmetric, the matrix $[\mu] = (t_{ij})$ shows the respective properties,

1'. It is $t_{i,i} = 1$, for all $1 \leq i \leq n$, that is, the main diagonal of $[\mu]$ is constituted by $n$ numbers equal to 1.

2'. It is $t_{ij} = t_{ji}$, for all $1 \leq i, j \leq n$, that is, the elements of $[\mu]$ are placed symmetrically with respect to the main diagonal.

For example, the matrix

$$
\begin{pmatrix}
1 & 0.7 \\
0.6 & 1
\end{pmatrix}
$$

is reflexive, but not symmetric,

and the matrix

$$
\begin{pmatrix}
1 & 0.6 & 0.7 \\
0.6 & 0 & 0.9 \\
0.7 & 0.9 & 0.5
\end{pmatrix}
$$

is symmetric, but not reflexive.

For what concerns the antisymmetric property, $t_{ij} > 0$ and $t_{ji} > 0$, implies $i = j$. For example, the matrix

$$
\begin{pmatrix}
1 & 0 & 0.7 & 0 \\
0.6 & 1 & 0 & 0.7 \\
0 & 0.5 & 1 & 0.8 \\
0.7 & 0 & 0 & 1
\end{pmatrix}
$$
4.3. WHICH RELEVANT PROPERTIES DO HAVE A FUZZY BINARY RELATION?

is antisymmetric.

Let’s define the binary relation, \([\mu] \leq [\sigma]\), between \(n \times n\) matrices if \(t_{ij} \leq s_{ij}\) for all \(1 \leq i, j \leq n\), provided \([\mu] = (t_{ij}), [\sigma] = (s_{ij})\). With such definition,

- \([\mu]\) reflects a \(T\)-transitive fuzzy relation \(\mu\), if and only if, \([\mu] \otimes_T [\mu] \leq [\mu]\).

The proof is as follows.

a. If \(\mu\) is \(T\)-transitive, from

\[
T(\mu(x_i, x_j), \mu(x_j, x_k)) \leq \mu(x_i, x_k),
\]

or \(T(t_{ij}, t_{jk}) \leq t_{ik}\), it is \(\text{Max}_{1 \leq j \leq n} T(t_{ij}, t_{jk}) \leq t_{ik}\). That is, \([\mu] \otimes_T [\mu] \leq [\mu]\).

b. If \([\mu] \otimes_T [\mu] \leq [\mu]\), or \(\text{Max}_{1 \leq j \leq n} T(t_{ij}, t_{jk}) \leq t_{ik}\), follows \(T(t_{ij}, t_{jk}) \leq t_{ik}\), for all \(1 \leq i, j \leq n\). That is, \(\mu\) is \(T\)-transitive.

Finally,

- If \(\mu\) is reflexive and \(T\)-transitive, it is \([\mu] \otimes_T [\mu] = [\mu]\). since

\[
t_{ik} = T(1, t_{ik}) = T(t_{ii}, t_{ik}) \leq \text{Max}_{1 \leq j \leq n} T(t_{ij}, t_{jk}) \leq t_{ik},
\]

implies

\[
t_{ik} = \text{Max}_{1 \leq j \leq n} T(t_{ij}, t_{jk}), \text{ or } [\mu] = [\mu] \otimes_T [\mu].
\]

Remarks 4.3.1. The definitions given in this section contain the case of the corresponding classical crisp definitions,

- If \(R \subseteq X \times X\) is a classical reflexive relation in \(X\), its membership function \(\mu_R\) reflects \((x, x) \in R\) for all \(x \in X\), by \(\mu_R(x, x) = 1\).
• If $R \subseteq X \times X$ is a classical symmetric relation in $X$,
  \[(x, y) \in R \iff (y, x) \in R\]
is reflected by $\mu(x, y) = \mu(y, x)$.

• If $R \subseteq X \times X$ is antisymmetric,
  \[(x, y) \in R \land (y, x) \in R \iff x = y\]
is reflected by $\mu_R(x, y) = \mu_R(y, x) = 1 > 0 \implies x = y$.

• If $R$ is transitive, $(x, y) \in R \land (y, z) \in R \implies (x, z) \in R$, is reflected by
  $\mu_R(x, y) = \mu_R(y, z) = 1 \implies \mu_R(x, z) = 1$, that implies,
  \[T(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z).\]
  Notice that if $\mu_R(x, y) = 0$ or $\mu_R(y, z) = 0$, then, for example,
  \[T(\mu_R(x, y), \mu_R(y, z)) = T(0, \mu_R(y, z)) = 0 \leq \mu_R(x, z).\]
  Hence, for all $x, y, z$ in $X$, and any continuous t-norm $T$, is
  \[T(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z).\]
  that reflects equationally the transitivity of $R$.

Example 4.3.2. 1. The matrix

\[
[\mu] = \begin{pmatrix}
1 & 1/8 & 2/8 \\
1/8 & 1 & 3/8 \\
2/8 & 3/8 & 1
\end{pmatrix}
\]
is reflexive and symmetric (fuzzy similarity). In addition,

\[
[\mu] \otimes_{\text{prod}} [\mu] = [\mu],
\]
that is, $\mu$ is prod-transitive. Hence, $\mu$ is a prod-indistinguishability.

Notice that,

\[
[\mu] \otimes_{\text{min}} [\mu] = \begin{pmatrix}
1 & 1/8 & 2/8 \\
1/8 & 1 & 3/8 \\
2/8 & 2/8 & 1
\end{pmatrix} \neq [\mu],
\]
and μ is not min-transitive. Of course, since \( W \preceq \text{prod} \), μ is also \( W \)-transitive. Notice that this last matrix is reflexive, non symmetric, but min-transitive, since

\[
\begin{pmatrix}
1 & 1/8 & 2/8 \\
1/8 & 1 & 3/8 \\
2/8 & 2/8 & 1
\end{pmatrix} \otimes_{\min}
\begin{pmatrix}
1 & 1/8 & 2/8 \\
1/8 & 1 & 3/8 \\
2/8 & 2/8 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 2/8 & 2/8 \\
2/8 & 1 & 3/8 \\
2/8 & 2/8 & 1
\end{pmatrix},
\]

hence, this \( 3 \times 3 \)-matrix reflects a min-preorder and, consequently, a \( T \)-preorder for all t-norm \( T \).

2. The before mentioned fuzzy relation \( \mu(x, y) = \max(0, 1 - k|x - y|) \) is not only reflexive and symmetric, but also \( W \)-transitive, as it can be proved by distinguishing the four cases: \( \frac{1}{k} \geq |x - y|, \frac{1}{k} \geq |y - z|; \frac{1}{k} < |x - y|, \frac{1}{k} < |y - z|; \frac{1}{k} \geq |x - y|, \frac{1}{k} < |y - z|; \) and \( \frac{1}{k} < |x - y|, \frac{1}{k} < |y - z| \). Hence, \( \mu \) is a \( W \)-indistinguishability.

### 4.4 The concept of T-state

Given a fuzzy relation \( \mu : X \times X \rightarrow [0, 1] \) and a continuous t-norm \( T \), a fuzzy set \( \sigma : X \rightarrow [0, 1] \) is a \( T \)-state of \( \mu \), if

\[
T(\sigma(x), \mu(x, y)) \leq \sigma(y), \forall (x, y) \in X \times X.
\]

All constant fuzzy sets \( \mu_k(x) = k \) for all \( x \in X \), and \( k \in [0, 1] \), are T-states of any fuzzy relation \( \mu : X \times X \rightarrow [0, 1] \): \( T(\mu_k(x), \mu(x, y)) \leq \mu_k(x) = k = \mu_k(y) \). For example, \( \mu_0 = \mu_{\varnothing} \) and \( \mu_1 = \mu_X \), are always \( T \)-states. Hence, the set \( T(\mu) \) of all \( T \)-states \( \sigma \) of \( \mu \) is never empty. From now on, in general we will only refer to non constant \( T \)-states \( \sigma \).

Given a fuzzy relation \( \mu : X \times Y \rightarrow [0, 1] \), once \( y \in Y \) is fixed, we can define the fuzzy set \( \mu_y : X \times X \rightarrow [0, 1] \), defined by,

\[
\mu_y(x) = \mu(x, y), \text{ for all } x \in X.
\]
When $X, Y$ are finite sets, $\mu_y$ is the y-column of the matrix $[\mu]$.

If $\mu : X \times X \rightarrow [0, 1]$ is a symmetric and $T-$transitive fuzzy relation, from

$$T(\mu(x, y), \mu(y, z)) \leq \mu(x, z), \text{ for all } x, y, z \text{ in } X,$$

follows $T(\mu(y, x), \mu(y, z)) \leq \mu(z, x)$, or

$$T(\mu_x(y), \mu(y, z)) \leq \mu_x(z),$$

that is, $\mu_x$ is a $T-$state of $\mu$.

For example, with $X = \{x_1, x_2, x_3\}$ and $\mu : X \times X \rightarrow [0, 1]$ given by

$$[\mu] = \begin{pmatrix} 1 & 1/8 & 2/8 \\ 1/8 & 1 & 3/8 \\ 2/8 & 3/8 & 1 \end{pmatrix}$$

it is $\mu_{x_1} = 1/x_1 + 1/8/x_2 + 2/8/x_3$, and since $\mu$ is a prod—indistinguishability, it results,

$$\mu_{x_1}(x_1)\mu(x_1, y) = \mu(x_1, y) = \mu_{x_1}(y),$$

it is also $\mu_{x_2} = 1/8/x_1 + 1/x_2 + 2/8/x_3$, and

$$\mu_{x_2}(x_1)\mu(x_1, y) = \frac{1}{8}\mu(x_1, y),$$

showing,

- $\frac{1}{8}\mu(x_1, x_1) = \frac{1}{8} = \mu_{x_2}(x_1)$
- $\frac{1}{8}\mu(x_1, x_2) = \frac{1}{8^2} < \frac{1}{8} = \mu_{x_2}(x_2)$
- $\frac{1}{8}\mu(x_1, x_3) = \frac{2}{8^2} < \frac{3}{8} = \mu_{x_2}(x_3)$,

etc. That is, the three fuzzy sets $\mu_{x_1}, \mu_{x_2}, \mu_{x_3}$ are prod—states of $\mu$.

Remark 4.4.1. When the fuzzy relation $\mu$ represents a conditional statement $Q/P$ (a fuzzy rule ‘If $x$ is $P$, then $y$ is $Q$’), the $T-$states of $\mu$ are among the fuzzy sets verifying the Modus Ponens with respect to the continuous t-norm $T$. 


4.5 Fuzzy relations and $\alpha$-cuts

Given a fuzzy relation $\mu : X \times X \to [0, 1]$, the $\alpha$-cuts of $\mu$ are the classical relations $\mu_\alpha$ defined by,

$$
\mu_\alpha = \{(x, y) \in X \times X; \mu(x, y) \geq \alpha\},
$$

for all $\alpha \in [0, 1]$. Obviously, $\mu(0) = X \times X$, and if $\alpha_1 \geq \alpha_2$, it is $\mu(\alpha_1) \subseteq \mu(\alpha_2)$.

- $\mu$ is symmetric, if and only if all its $\alpha$-cuts are symmetric (classical) relations.

$$
\mu(x, y) = \mu(y, x) \iff (x, y) \in \mu(\alpha) \text{ and } (y, x) \in \mu(\alpha).
$$

- $\mu$ is reflexive, if and only if all $\alpha$-cuts are reflexive (classical) relations.

$$
\mu(x, x) = 1 \iff (x, x) \in \mu(\alpha), \text{ since } \alpha \leq 1.
$$

- If $\mu$ is antisymmetric, all the $\alpha$-cuts are antisymmetric (crisp) relations,

If $\mu(x, y) \geq \alpha > 0$, and $\mu(y, x) \geq \alpha > 0$, it is $x = y$.

- If $\mu$ is a $T$-transitive fuzzy relation,

$$
(x, y) \in \mu(\alpha), \text{ and } (y, z) \in \mu(\alpha), \text{ implies } (x, z) \in \mu(T(\alpha, \alpha)),
$$

since,

$$
\mu(x, y) \geq \alpha, \mu(y, z) \geq \alpha \Rightarrow T(\alpha, \alpha) \leq T(\mu(x, y)\mu(y, z)) \leq \mu(x, z).
$$

Hence, only if is $\mu$ is min-transitive, it is sure that all $\alpha$-cuts are classical preorders, and $\mu$ results decomposed in the family of preorders $\{\mu_\alpha; \alpha \in (0, 1)\}$. 
Example 4.5.1. With $X = \{1, 2, 3, 4\}$, the matrix

$$[\mu] = \begin{pmatrix}
1 & 0.6 & 1 & 0.6 \\
0.3 & 1 & 0.3 & 0.3 \\
1 & 0.6 & 1 & 0.6 \\
0.4 & 0.8 & 0.4 & 1
\end{pmatrix},$$

is obviously reflexive but not symmetric, and verifies $[\mu] \otimes_{\text{min}} [\mu] = [\mu]$. Hence, $\mu$ is a min-preorder. Its different $\alpha$-cuts are

$$[\mu_{(1)}] = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad [\mu_{(0.8)}] = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}, \quad [\mu_{(0.6)}] = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix},$$

$$[\mu_{(0.4)}] = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}, \quad [\mu_{(0.3)}] = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix},$$

that give the classical $\leq_{(\alpha)}$ preorders, that follows:

- $\leq_{(1)}$: $1 \leq_{(1)} 1$, $2 \leq_{(1)} 2$, $3 \leq_{(1)} 3$, $4 \leq_{(1)} 4$, $1 \leq_{(1)} 3$, $3 \leq_{(1)} 1$.
- $\leq_{(0.8)}$: $1 \leq_{(0.8)} 1$, $2 \leq_{(0.8)} 2$, $3 \leq_{(0.8)} 3$, $4 \leq_{(0.8)} 4$, $1 \leq_{(0.8)} 3$, $3 \leq_{(0.8)} 1$, $4 \leq_{(0.8)} 2$.
- $\leq_{(0.6)}$: $1 \leq_{(0.6)} 1$, ..., $4 \leq_{(0.6)} 4$, $1 \leq_{(0.6)} 3$, $3 \leq_{(0.6)} 1$, $4 \leq_{(0.6)} 2$, $1 \leq_{(0.6)} 4$, $3 \leq_{(0.6)} 2$, $3 \leq_{(0.6)} 4$.
- $\leq_{(0.4)}$: $1 \leq_{(0.4)} 1$, ..., $4 \leq_{(0.4)} 4$, $1 \leq_{(0.4)} 3$, ..., $4 \leq_{(0.4)} 1$, $4 \leq_{(0.4)} 3$.
- $\leq_{(0.3)}$: $1 \leq_{(0.3)} 1$, ..., $4 \leq_{(0.3)} 3$, $2 \leq_{(0.3)} 1$, ..., $4 \leq_{(0.3)} 2$.

and, since, $0.3 \leq 0.4 \leq 0.6 \leq 0.8 \leq 1$, verify $\leq_{(1)} \subseteq \leq_{(0.8)} \subseteq \leq_{(0.6)} \subseteq \leq_{(0.4)} \subseteq \leq_{(0.3)}$. 


Example 4.5.2. The fuzzy relation \( \mu : X \times X \to [0, 1] \), with \( X = \{1, 2, 3, 4, 5, 6\} \), given by

\[
[\mu] = \begin{pmatrix}
1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\
0.2 & 1 & 0.2 & 0.8 & 0.2 \\
1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\
0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\
0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.8 \\
0.6 & 0.2 & 0.6 & 0.8 & 0.8 & 1
\end{pmatrix}
\]

is reflexive, symmetrical and min-transitive, since \([\mu] \otimes_{\min} [\mu] = [\mu]\). Hence, all the \( \alpha \)-cuts are classical equivalence relations, each one defining a partition of \( X \). The different values of \( \alpha \) are 0.2, 0.6, 0.8 and 1 (levels of crispness of \( \mu \)), and it is easy to see that the corresponding partitions \( \pi_{0.2} \), \( \pi_{0.6} \), \( \pi_{0.8} \), and \( \pi_{1} \), can be located as the partition tree:

![Partition Tree]

This tree is called the fuzzy quotient of \( X \) by \( \mu \).

Example 4.5.3. The fuzzy relation \( \mu : \{1, 2, ..., 6\} \to [0, 1] \) given by
CHAPTER 4. FUZZY RELATIONS

$$\mu = \begin{pmatrix}
1 & 0.8 & 0.2 & 0.6 & 0.6 & 0.4 \\
0 & 1 & 0 & 0 & 0.6 & 0 \\
0 & 0 & 1 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 1 & 0.6 & 0.4 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$

is reflexive, antisymmetric and min-transitive. Hence, is a fuzzy ordering whose $\alpha$-cuts should be crisp partial orderings. Namely,

1. $[\mu_{(1)}] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},$ with partial order

that only connects the pairs \((1,1), (2,2), ... (6,6)\), and is called the disconnected partial order.

2. $[\mu_{(0.8)}] = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$

3. $[\mu_{(0.6)}] = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$
4.5. FUZZY RELATIONS AND $\alpha$-CUTS

4. $[\mu_{(0.5)}]$ =
\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

5. $[\mu_{(0.4)}]$ =
\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

All this crisp partial orderings come from the fuzzy ordering given by $[\mu]$, and which valued directed graph is

\[
\begin{pmatrix}
2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \\
1 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \\
1 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \\
1 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \\
1 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \\
1 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \\
\end{pmatrix}
\]

where some arrows are avoided because of the min-transitivity of $\mu$. For instance, the degree between 1 and 5, is

\[
\mu(1, 5) = \text{Max}_{1 \leq k \leq 6} \text{min}(\mu(1, k), \mu(k, 5)) = \text{max}(\text{min}(0.8, 0.6), \text{min}(0.2, 0.5)) = 0.6,
\]

and the strength of the order between 1 and 6, is

\[
\mu(1, 6) = \text{Max}_{1 \leq k \leq 6} \text{min}(\mu(1, k), \mu(k, 6)) = \text{min}(0.6, 0.4) = 0.4.
\]
Of course, $\mu(1, 1) = \mu(2, 2) = \ldots = \mu(6, 6) = 1$. 
Chapter 5

T-PREORDERS AND T-INDISTINGUISHABILITIES

5.1 Which is the aim of this section?

To characterize the T-Preorders and the T-indistinguishabilities relations by means of particles classes of them, and to give ways of constructing T-preorders and T-indistinguishabilities.

Given a fuzzy relation on $X, \mu : X \times X \rightarrow [0, 1]$, with the three properties

1. Reflexive, $\mu(x, x) = 1$, for all $x \in X$

2. Symmetry, $\mu(x, y) = \mu(y, x)$, for all $(x, y) \in X \times X$

3. T-transitivity, $T(\mu(x, y), \mu(y, z)) \leq \mu(x, z)$, for all $x, y, z$ in $X$, and a continuous t-norm $T$,

it can be named

- T-Preorders, those $\mu$ verifying 1 and 3.

- T-Indistinguishabilities, those $\mu$ verifying 1, 2, and 3.
• Similarities, those $\mu$ verifying 1 and 2.

A particular class of T-Preorders is given by the known operators (R-implications)

$$J_T(a, b) = \text{Sup}\{z \in [0, 1]; T(z, a) \leq b\},$$

and the particular class of T-indistinguishabilities is given by the operators

$$E_T(a, b) = \text{Min}(J_T(a, b), J_T(b, a)),$$

shortly written $E_T = \min(J_T, J_T^3)$, with $J_T^3(a, b) = J_T(b, a)$. It is obvious that

$$J_T(a, a) = 1,$$

and that $J_T(a, b) \neq J_T(b, a)$,

as well as that $E_T(a, a) = 1$, and $E_T(a, b) = E_T(b, a)$. What it is not so obvious is that the relations $J_T$ are T-Transitive:

$$T(J_T(a, b), J_T(b, c)) \preceq J_T(a, c) \text{ for all } a, b, c \in [0, 1]$$

To avoid some difficult technicalities, we will just exemplify this general result in the particular case $J_W(a, b) = \min(1, 1-a+b)$:

$$W(J_W(a, b), J_W(b, c)) = W(\min(1, 1-a+b), \min(1, 1-b+c)) = \max(0, \min(0, b-a) + \min(1, 1-b+c)) =
$$

$$= \begin{cases} 
\min(1, 1-b+c) \leq (1, 1-a+c), & \text{if } a > b \\
\max(0, b-a + \min(1, 1-b+c)) \leq \min(1, 1-a+c), & \text{if } a \leq b 
\end{cases} \leq J_W(a, c).$$

with the T-transitivity of $J_T$, it is

$$T(E_T(a, b), E_T(b, c)) = T(\min(J_T(a, b), J_T(b, a)), \min(J_T(b, c), J_T(c, b))) \leq$$

$$\leq T(J_T(a, b), J_T(b, c)) = E_T(a, c),$$

hence, all relations $E_T$ are T-transitive. Then,

- All relations $J_T$ are T-Preorders, and
- All relations $E_T$ are T-Indistinguishabilities
5.2. THE CHARACTERIZATION OF T-PREORDERS

• If \( \{ R_i; i \in I \} \) is a collection of T-Preorders, \( \inf_{i \in I} R_i(x, y) = R(x, y) \), is also a T-Preorder. Obviously, \( R(x, x) = \inf_{i \in I} R_i(x, x) = 1 \), and

\[
T(R(a, b), R(b, c)) = T(\inf_{i \in I} R_i(a, b), \inf_{i \in I} R_i(b, c)) \leq
\]

\[
\inf_{i \in I} T(R_i(a, b), R_i(b, c)) \leq \inf_{i \in I} R_i(a, c) = R(a, c),
\]

since \( T \) is a continuous t-norm. \( \Box \)

• If \( \{ E_i; i \in I \} \) is a collection of T-indistinguishabilities, \( \inf_{i \in I} E_i(x, y) = E(x, y) \) is also a T-indistinguishability. Obviously,

\[
E(a, a) = \inf_{i \in I} E_i(a, a) = 1,
\]

and

\[
E(a, b) = \inf_{i \in I} E_i(a, b) = \inf_{i \in I} E_i(b, a) = E(b, a).
\]

Finally,

\[
T(E(a, b), E(b, c)) = T(\inf_{i \in I} E_i(a, b), \inf_{i \in I} E_i(b, c)) \leq
\]

\[
\inf_{i \in I} T(E_i(a, b), E_i(b, c)) \leq \inf_{i \in I} E_i(a, c) = E(a, c)
\]

since \( T \) is a continuous t-norm. \( \Box \)

5.2 The characterization of T-Preorders

If \( \sigma \in [0, 1]^X \), define \( J_T^\sigma(x, y) = J_T(\sigma(x), \sigma(y)) \). Obviously \( J_T^\sigma \) is a T-Preorder in \( X \times X \), and if \( \mathcal{F} \subset [0, 1]^X \) it is also

\[
\inf_{\sigma \in \mathcal{F}} J_T^\sigma
\]

a T-Preorder.
CHAPTER 5. T-PREORDERS AND T-INDISTINGUISHABILITIES

Given a fuzzy relation \( \mu : X \times X \rightarrow [0, 1] \), consider the set \( T(\mu) \) of its T-states, that is the fuzzy sets \( \sigma : X \rightarrow [0, 1] \), such that

\[
T(\sigma(x), \mu(x, y)) \leq \sigma(y), \text{ for all } x, y \in X.
\]

As it is known this last inequality is equivalent to

\[
\mu(x, y) \leq J_T(\sigma(x), \sigma(y)) = J_T^\sigma(x, y),
\]

thus,

\[
\sigma \in T(\mu) \iff \mu \leq J_T^\sigma.
\]

Hence,

\[
\sigma \in T(\mu) \iff \mu \leq \inf_{\sigma \in T(\mu)} J_T^\sigma,
\]

and it is clear that if \( \mu = \inf_{\sigma \in T(\mu)} J_T^\sigma \), \( \mu \) is a T-Preorder. Avoiding some technical difficulties in the proof of the converse statement, let us state

- \( \mu \) is a T-Preorder if and only if \( \mu = \inf_{\sigma \in T(\mu)} J_T^\sigma \),

a result characterizing the structure of all T-preorders.

For example,

- \( T = \text{min} \), all min-preorders are of the form,

\[
\mu(x, y) = \inf_{\sigma \in T(\mu)} J_{\text{min}}^\sigma = \inf_{\sigma \in T(\mu)} \begin{cases} 
1, & \text{if } \sigma(x) \leq \sigma(y) \\
\sigma(y), & \text{if } \sigma(x) > \sigma(y)
\end{cases}
\]

- \( T = \text{prod}_\varphi \), all \( \text{prod}_\varphi \)-preorders are of the form,

\[
\mu(x, y) = \inf_{\sigma \in T(\mu)} J_{\text{prod}_\varphi}^\sigma = \inf_{\sigma \in T(\mu)} \begin{cases} 
1, & \text{if } \sigma(x) \leq \sigma(y) \\
\varphi^{-1}(\varphi(y)/\varphi(x)), & \text{if } \sigma(x) > \sigma(y)
\end{cases}
\]

- \( T = W_\varphi \), all \( W_\varphi \)-preorders are of the form

\[
\mu(x, y) = \inf_{\sigma \in T(\mu)} J_W^\sigma(x, y) = \inf_{\sigma \in T(\mu)} \min(1, 1 - \sigma(x) + \sigma(y))
\]
5.3. THE CHARACTERIZATION OF T-INDISTINGUISHABILITIES

Remark 5.2.1. As an immediate consequence of all that has been said, for any family $\mathcal{F} \subseteq [0,1]^X$, the T-preorder $\text{Inf}_T^\sigma$ has the elements in $\mathcal{F}$ as T-states. If $\mu(x,y) = \text{Inf}_T^\sigma \forall \sigma \in \mathcal{F}$, it follows $\mu \leq J_T^\sigma$, or, equivalently $\sigma \in T(\mu), \forall \sigma \in \mathcal{F}$, or $\mathcal{F} \subseteq T(\mu)$. For example, with $X = [0,1]$ and the two functions $\sigma_1(x) = x, \sigma_2(x) = 1 - x$, it is

$$
\mu(x,y) = \text{Min}(J_T^{\sigma_1}(x,y), J_T^{\sigma_2}(x,y))
$$

a T-preorder. With $T = W$,

$$
\mu(x,y) = \text{min}(\text{min}(1,1 - \sigma_1(x) + \sigma_1(y)), \text{min}(1,1 - \sigma_2(x) + \sigma_2(y))) = \\
\text{min}(1,1 - x + y, 1 + x - y),
$$

is a W-preorder.

5.3 The characterization of T-indistinguishabilities

Let us consider, the T-indistinguishabilities

$$
E_T^\sigma(x,y) = \text{min}(J_T^\sigma(x,y), J_T^\sigma(y,x)) = \text{min}(J_T(\sigma(x), \sigma(y)), J_T(\sigma(y), \sigma(x))).
$$

If $\mu$ is a fuzzy relation in $[0,1]^{X \times X}$, take $T(\mu)$. Obviously

$$
\mu(x,y) \leq \text{Inf}_{\sigma \in T(\mu)} E_T^\sigma(x,y), \text{ for all } x,y \in X.
$$

In the same vein that in the case of T-Preorders,

- $\mu$ is a T-indistinguishability if and only if $\mu = \text{Inf}_{\sigma \in T(\mu)} T^\sigma$.

For example,

- $T = \text{min}, \mu(x,y) = \text{Inf}_{\sigma \in T(\mu)} \text{min} \left\{ \begin{array}{ll} 1, & \text{if } \sigma(x) \leq \sigma(y) \\ \sigma(y), & \text{if } \sigma(x) > \sigma(y) \end{array} \right\} = \\
\text{Inf}_{\sigma \in T(\mu)} \left\{ \begin{array}{ll} 1, & \text{if } \sigma(x) = \sigma(y) \\ \text{min}(\sigma(x), \sigma(y)), & \text{otherwise} \end{array} \right\} = \\
\text{Inf}_{\sigma \in T(\mu)} \left\{ \begin{array}{ll} 1, & \text{if } \sigma(x) = \sigma(y) \\ \text{min}(\sigma(x), \sigma(y)), & \text{otherwise} \end{array} \right\}$
CHAPTER 5. T-PREORDERS AND T-INDISTINGUISHABILITIES

- \( T = \text{prod}, \mu(x,y) = \inf_{\sigma \in T(\mu)} \left( \begin{array}{c} 1, \quad \text{if } \sigma(x) \leq \sigma(y) \\ \frac{\sigma(y)}{\sigma(x)}, \quad \text{if } \sigma(x) > \sigma(y) \end{array} \right) \)

\[
\inf_{\sigma \in T(\mu)} \begin{cases} 1, & \text{if } \sigma(x) = \sigma(y) \\ \min \left( \frac{\sigma(y)}{\sigma(x)}, \frac{\sigma(x)}{\sigma(y)} \right), & \text{otherwise} \\ 1, & \text{if } \sigma(x) = \sigma(y) \end{cases} = \left\{ \begin{array}{c} 1, \quad \text{if } \sigma(x) = \sigma(y) \\ \inf_{\sigma \in T(\mu)} \min \left( \frac{\sigma(y)}{\sigma(x)}, \frac{\sigma(x)}{\sigma(y)} \right), \quad \text{otherwise} \end{array} \right.
\]

- \( T = W, \mu(x,y) = \inf_{\sigma \in T(\mu)} \min(1, 1 - \sigma(x) + \sigma(y), 1 - \sigma(y) + \sigma(x)) = \inf_{\sigma \in T(\mu)} \min(1, 1 - \max(\sigma(x) - \sigma(y), \sigma(y) - \sigma(x))) = \inf_{\sigma \in T(\mu)} (1 - |\sigma(x) - \sigma(y)|) \)

Remark 5.3.1. Like in the case of T-Preorders, for any family of functions \( \mathcal{F} \subset [0,1]^X \), the T-indistinguishability \( \inf E_T^\mu \) has the elements in \( \mathcal{F} \) as T-states. Notice that with \( \mu(x,y) = \inf_{\sigma \in \mathcal{F}} E_T^\sigma (x,y) = \inf_{\sigma \in \mathcal{F}} \min(J_T^\sigma (x,y), J_T^\sigma (y,x)), \) results \( \mu(x,y) \leq J_T^\sigma (x,y), \) for all \( x,y \in X, \) that is equivalent to \( \sigma \in T(\mu). \)

Hence,

\[ \forall \sigma \in \mathcal{F} \Rightarrow \sigma \in T(\mu), \text{ or } \mathcal{F} \subset T(\mu). \]

With \( X = [0,1] \) and the two functions \( \sigma_1(x) = x, \sigma_2(x) = 1 - x, \) it is

\[ \mu(x,y) = \min(E_T^{\sigma_1}(x,y), E_T^{\sigma_2}(x,y)), \]

a T-indistinguishability.

- With \( T = W, \) is

\[ \mu(x,y) = \min(1 - |\sigma_1(x) - \sigma_1(y)|, 1 - |\sigma_2(x) - \sigma_2(y)|) = \]

\[ = \min(1 - |x - y|, 1 - |y - x|) = 1 - |x - y|. \]

Notice that with \( \sigma_1(x) = x, \sigma_2(x) = x^2, \) results

\[ \mu(x,y) = \min(1 - |x - y|, 1 - |x^2 - y^2|), \]

that is, \( \mu(x,y) = 1 - |x - y|, \) provided \( x + y \leq 1, \) and \( \mu(x,y) = 1 - |x^2 - y^2| \)

if \( x + y > 1. \)
5.3. THE CHARACTERIZATION OF T-INDISTINGUISHABILITIES

- With $T = \text{prod}$, results

$$
\mu(x, y) = \text{Min} \left( \begin{cases} 
1, & x \leq y \\
\frac{y}{x}, & x > y
\end{cases} \right) = \text{Min} \left( \begin{cases} 
1, & y \leq x \\
\frac{1-y}{x}, & y > x
\end{cases} \right) = \begin{cases} 
1, & x = y \\
\frac{y}{x}, & y < x \\
\frac{1-y}{x}, & y > x
\end{cases}
$$

- With $T = \text{min}$, results

$$
\mu(x, y) = \text{min} \left( \begin{cases} 
1, & x \leq y \\
y, & x > y
\end{cases} \right) = \text{min} \left( \begin{cases} 
1, & y \leq x \\
1-y, & y > x
\end{cases} \right) = \begin{cases} 
1, & x = y \\
y, & x \neq y \leq 1/2 \\
1-y, & x \neq y > 1/2
\end{cases}
$$

Example 5.3.2. A finite example

With $X = \{1, 2, 3, 4\}$, take $\sigma_1(x) = \frac{x}{4}$, $\sigma_2(x) = 1 - \frac{x}{4}$, and $T = W$. It is

$$
[J^\sigma_1_W] = \begin{pmatrix}
1 & 1 & 1 & 1 \\
3/4 & 1 & 1 & 1 \\
1/2 & 3/4 & 1 & 1 \\
1/4 & 1/2 & 3/4 & 1
\end{pmatrix}
$$

Since, f.ex.,

$$
J^\sigma_1_W(1, 2) = \text{min}(1, 1 - \frac{1}{4} + \frac{2}{4}) = 1, \quad J^\sigma_1_W(3, 1) = \text{min}(1, 1 - \frac{3}{4} + \frac{1}{4}) = 1/2,
$$

$$
J^\sigma_1_W(4, 1) = \text{min}(1, 1 - 1 + \frac{1}{4}) = 1/4, \quad J^\sigma_1_W(3, 4) = \text{min}(1, 1 - \frac{3}{4} + 1) = 1, \text{ etc.}
$$

It is also

$$
[J^\sigma_2_W] = \begin{pmatrix}
1 & 3/4 & 1/2 & 1/4 \\
1 & 1 & 3/4 & 1/2 \\
1/2 & 1 & 1 & 3/4 \\
1 & 1 & 1 & 1
\end{pmatrix}
$$

since, f.ex.,

$$
J^\sigma_2_W(1, 2) = \text{min}(1, 1 - \frac{3}{4} + \frac{1}{2}) = 3/4, \quad J^\sigma_2_W(3, 1) = \text{min}(1, 1 - \frac{1}{4} + \frac{3}{4}) = 1/2,
$$
\[ J_W^2(4, 1) = \min(1, 1 - 0 + \frac{3}{4}) = 1, \quad J_W^2(3, 4) = \min(1, 1 - \frac{1}{4} + 0) = 3/4, \text{ etc.} \]

Hence,

\[
\begin{bmatrix}
E_W
\end{bmatrix} = \min([J_W^1], [J_W^2]) = \begin{pmatrix}
1 & 3/4 & 1/2 & 1/4 \\
3/4 & 1 & 3/4 & 1/2 \\
1/2 & 3/4 & 1 & 3/4 \\
1/4 & 1/2 & 3/4 & 1 \\
\end{pmatrix}
\]

is a \(W\)-indistinguishability on \(X\), with the directed graph

---

**Example 5.3.3.** Last example

With an intersection \(\mu \cdot \sigma = T \circ (\mu \times \sigma)\), \((T\) a continuous t-norm) define

\[
E_T(\mu, \sigma) = \text{Sup} (\mu \cdot \sigma).
\]

Obviously, \(E_T(\mu, \sigma) = E_T(\sigma, \mu)\), since \(T\) is a commutative operation. To have,

\[
E_T(\mu, \mu) = 1, \text{ or } \text{Sup} T \circ (\mu \times \mu) = 1,
\]
5.3. THE CHARACTERIZATION OF T-INDISTINGUISHABILITIES

since $T \circ (\mu \times \mu) \leq \mu$, it should be $\Sup \mu = 1$, that is, since $X$ is a finite set, $\mu$ should be a normalized fuzzy set, a fuzzy set for which it exists $x_0 \in X$ such that $\mu(x_0) = 1$.

From now on let’s consider the set $\mathcal{N}(X) = \{\mu \in [0, 1]^X; \Sup \mu = 1\}$. The mapping

$$E_T : \mathcal{N}(X) \times \mathcal{N}(X) \rightarrow [0, 1],$$

is reflexive and symmetric, that is, $E_T$ is a similarity in $\mathcal{N}(X)$. Obviously, $E_T \leq E_{\min}$, for all continuous t-norm $T$. Since,

$$T(E_T(\mu, \sigma), E_T(\sigma, \lambda)) = T(\Sup T(\mu, \sigma), \Sup T(\sigma, \lambda)) = \Sup T(T(\mu, \sigma), T(\sigma, \lambda)) \leq \Sup T(\mu, \lambda) = E_T(\mu, \lambda),$$

it results that, in addition, $E_T$ is a $T^*$-indistinguishability for all continuous t-norm $T^*$ such that $T^* \leq T$. In particular, $E_{\min}$ is not only a min-indistinguishability but a $T$-indistinguishability for all continuous t-norm $T$.

For example with $X = \{1, 2, 3, 4\}$, and the two fuzzy sets

$$\mu = 1|1 + 0.4|2 + 0.8|3 + 0.7|4, \sigma = 0.6|1 + 0.5|2 + 0.8|3 + 1|4$$

it follows

$$\mu \cdot \sigma = 0.6|1 + T(0.4, 0.5)|2 + T(0.8, 0.8)|3 + 0.7|4,$$

and

- $E_{\min}(\mu, \sigma) = 0.8$, since $\min(0.8, 0.8) = 0.8$, $\min(0.4, 0.5) = 0.4$
- $E_{\prod}(\mu, \sigma) = 0.64$, since $\prod(0.8, 0.8) = 0.64$, $\prod(0.4, 0.5) = 0.2$
- $E_W(\mu, \sigma) = 0.6$, since $W(0.8, 0.8) = 0.6$, $W(0.4, 0.5) = 0$

**Remark 5.3.4. Last Remark**

Provided $T = \min$ or $T = \prod_\phi$, if $E_T(\mu, \sigma) > 0$, and $E_T(\sigma, \alpha) > 0$, it is

$$0 < T(E_T(\mu, \sigma), E_T(\sigma, \alpha)) \leq E_T(\mu, \alpha),$$
and $E_T(\mu, \alpha) > 0$. In these cases, $E_T$ is said to be strictly transitive.

Nevertheless, in the case in which a fuzzy relation $E : X \times X \to [0, 1]$ is $W_\varphi$-transitive, from

$$E(x, y) > 0 \text{ and } E(y, z) > 0,$$

what follows is that, from the equivalence given by the existence of $r > 0$ such that $E(x, y) > r$, $E(y, z) > r$, it is

$$W_\varphi(r, r) = \varphi^{-1}(\max(0, 2\varphi(r) - 1)) \leq W_\varphi(E(x, y), E(y, z)) \leq E(x, z).$$

Hence, to have $0 < E(x, z)$, it is necessary that $0 < W_\varphi(r, r)$, that is,

$$0 < \max(0, 2\varphi(r) - 1), \text{ or } r > \varphi^{-1}(0.5).$$

For example, if it is $T = W (\varphi = id)$, it should be $r > 0.5$, and if $T = W_\varphi$ with $\varphi(x) = x^2$ it should be $r > \sqrt{0.5} = 0.7071$. In the case of the fuzzy relation

$$E(\mu, \sigma) = \frac{\sum_{i=1}^{n} \min(\mu(x_i), \sigma(x_i))}{\max(\sum_{i=1}^{n} \mu(x_i), \sum_{i=1}^{n} \sigma(x_i))},$$

with $X = \{x_1, \ldots, x_n\}$, it results $W_\varphi$-transitive for $\varphi(x) = x^2$. Thus, if

$$0.71 < E(\mu, \sigma), \text{ and } 0.71 < E(\sigma, \alpha)$$

it follows $0 < E(\mu, \alpha)$, since $W_\varphi(0.71, 0.71) = \sqrt{\max(0, 2 \cdot 0.71^2 - 1)} = 0.09$. 


Chapter 6

Fuzzy arithmetic

6.1 Introduction

6.1.1

As it was explained before, any operation $*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, can be extended to one $\odot: [0, 1]^{\mathbb{R}} \times [0, 1]^{\mathbb{R}} \to [0, 1]^{\mathbb{R}}$, by means of the extension principle:

$$(\mu \odot \sigma)(t) = \sup_{t=x+y} \min(\mu(x), \sigma(y)).$$

This extension includes the crisp subsets $A \subset \mathbb{R}$, since $\mu_A \in \{0, 1\}^{\mathbb{R}} \subset [0, 1]^{\mathbb{R}}$. For example, with $A_1 = \{1, 2, 3\}$, and $A_2 = \{1, 3, 5\}$, and only taking into account the numbers in $\mathbb{N} \subset \mathbb{R}$, it is

$$(\mu_{A_1} \odot \mu_{A_2})(t) = \sup_{t=x+y} \min(\mu_{A_1}(x), \mu_{A_2}(y)), \text{ with } t, x, y \text{ in } \mathbb{N},$$

and + the addition of natural numbers. Since $x + y \in \{2, 4, 5, 3, 5, 7, 8\}$, it results $\mu_{A_2} \odot \mu_{A_2} = \mu_{\{2,3,4,5,6,7,8\}}$, that is $A_1 \oplus A_2 = \{2, 3, 4, 5, 6, 7, 8\}$.

In the same vein, if $A_1 = [a, b]$, and $A_2 = [c, d]$, it results

$$(\mu_{A_1} \oplus \mu_{A_2})(t) = \sup_{t=x+y} \min(\mu_{[a,b]}(x), \mu_{[c,d]}(y)) = \mu_{[a+c,b+d]}(t), \text{ or } [a, b] \oplus [c, d] = [a+c, b+d].$$

Analogously, it results
• \([a, b] \odot [c, d] = [a - d, b - c]\)

• \([a, b] \otimes [c, d] = [\min(ad, ac, bd, bc), \max(ad, ac, bd, bc)]\)

• If \(0 \notin [c, d]\), \([a, b] \oplus [c, d] = [\min(\frac{\alpha}{\varepsilon}, \frac{\beta}{\varepsilon}, \frac{\gamma}{\delta}, \frac{\eta}{\delta}), \max(\frac{\alpha}{\varepsilon}, \frac{\beta}{\varepsilon}, \frac{\gamma}{\delta}, \frac{\eta}{\delta})]\).}

For example,

• \([7, 8] \oplus [-1, 9] = [6, 17]\)

• \([7, 8] \ominus [-1, 9] = [-2, 9]\)

• \([3, 4] \odot [2, 2] = [6, 8]\)

• \([4, 10] \odot [1, 2] = [2, 10]\)

• \(2 \ominus [7, 8] = [2, 2] \oplus [7, 8] = [9, 10]\)

• \(2 \odot [7, 8] = [2, 2] \odot [7, 8] = [\min(14, 16, 14, 16), \max(14, 16)] = [14, 16]\)

• \([7, 8] \odot 2 = [7, 8] \odot [2, 2] = [\min(\frac{7}{2}, \frac{8}{3}), \max(\frac{7}{2}, \frac{8}{3})] = [\frac{7}{2}, 4]\)

\subsection{6.1.2}

For all \(\mu \in [0, 1]^R\), and all \(r \in [0, 1]\), it can be computed that:

• \((r \oplus \mu)(t) = (\mu_r \odot \mu)(t) = \sup_{t=x+y} \min(\mu_r(x), \mu(y)) = \\
= \sup_{t=x+y} \min \left(1, \mu(y), \begin{cases} \mu(y), & x = r \\ 0, & x \neq r \end{cases} \right) = \sup_{t=x+y} \left(\mu(y), \begin{cases} \mu(y), & x = r \\ 0, & x \neq r \end{cases} \right) = \\
= \mu(t - r). \text{ Hence } ((-1) \oplus \mu)(t) = \mu(t - 1), \ (\mu_1 \odot \mu)(t) = \mu(t - 1), \ (\mu_0 \odot \mu)(t) = \mu(t).\)

• \((r \odot \mu)(t) = (\mu_r \odot \mu)(t) = \sup_{t=x+y} \min(\mu_r(x), \mu(y)) = \mu(\frac{t}{r}), \text{ if } r \neq 0. \text{ Hence,} \\
(\frac{1}{r} \odot \mu)(t) = \mu(rt), \ (1 \odot \mu)(t) = \mu(t), \ (\mu_1 \odot \mu)(t) = \mu(t).\)

• \((\mu_0 \odot \mu)(t) = \sup_{t=x+y} \min(\mu_0(x), \mu(y)) = 0 = \mu_0(t).\)
6.1. INTRODUCTION

- \( \frac{1}{\mu}(t) = \sup_{t=\frac{1}{y}} \min(1, \mu(y)) = \sup_{t=\frac{1}{y}} \mu(y) = \begin{cases} \mu(\frac{1}{t}), & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases} \)

Example 6.1.1. Given the fuzzy set

compute \( 2 \odot \mu \) and \( 2 \oplus \mu \).

It is \( (2 \odot \mu)(t) = \mu(t/2) \), and \( (2 \oplus \mu)(t) = \mu(t - 2) \). Hence,

6.1.3

It should be pointed out that, since the cartesian product \( \mu \times \sigma \) of \( \mu \in [0, 1]^X \) and \( \sigma \in [0, 1]^Y \), is defined by

\[
\mu \times \sigma(x, y) = \min(\mu(x), \sigma(y)), \mu \times \sigma \in [0, 1]^{X \times Y},
\]

the operations \( \odot \), with \( * \in \{+,-,\cdot,:\} \), can also be represented in the form

\[
\mu \odot \sigma(t) = \sup_{t=x*y} (\mu \times \sigma)(x, y),
\]

that in the finite case are simply,

\[
\mu \odot \sigma(t) = \max_{t=x*y} (\mu \times \sigma)(x, y).
\]
Example 6.1.2. If \( X = \{1, 2, 3\} \), and \( \mu = 0.7[1+0.9]2+1[3, \sigma = 0.8[1+0.9]3 \), since \( X \times X = \{(1, 1), (1, 2), \ldots, (3, 2), (3, 3)\} \), it results

\[
\mu \times \sigma = 0.7|(1, 1) + 0|(1, 2) + 0.7|(1, 3) + 0.8|(2, 1) + \\
+0.1|(2, 2) + 0.9|(2, 3) + 0.8|(3, 1) + 0|(3, 2) + 0.9|(3, 3).
\]

Since \( t = x + y \) will take the values \( 1 + 1 = 2, 1 + 2 = 3, 1 + 3 = 4, 2 + 1 = 3, 2 + 2 = 4, 2 + 3 = 5, 3 + 1 = 4, 3 + 2 = 5, 3 + 3 = 6 \), that is \( t \in \{2, 3, 4, 5, 6\} \), it results:

\[
\mu \oplus \sigma(2) = 0.7, \mu \oplus \sigma(3) = max(0, 0.8) = 0.8, \mu \oplus \sigma(4) = max(0.7, 0.8, 0) = 0.8,
\]

\[
\mu \oplus \sigma(5) = max(0.9, 0) = 0.9, \mu \oplus \sigma(6) = 0.9
\]

Hence,

\[
\mu \oplus \sigma = 0.7[2 + 0.8]3 + 0.8[4 + 0.9]5 + 0.9[6.
\]

For the product \( \mu \odot \sigma \), it is \( t = x \cdot y \in \{1, 2, 3, 4, 6, 9\} \), and

\[
\mu \cdot \sigma = 0.7[1 + 0.8]2 + 0.8[3 + 0]4 + 0.9[6 + 0.9]9.
\]

### 6.2 Fuzzy numbers

#### 6.2.1

In the usual scientific computation is rather unusual to consider exact numbers, as in the case of taking \( \pi \approx 3.1416 \), or \( \sqrt{2} \approx 1.4142 \). In some cases, what in taken is an interval containing the number, for example, \( \sqrt{2} \in [1.412, 1.413] \). Fuzzy numbers are just a “fuzzification” of this last idea. For example, the fuzzy number “approximately 5” can be taken either as the fuzzy set,
or as the fuzzy set

That kind of fuzzy sets are known as *fuzzy numbers*. Their definition is the following.

**Definition 6.2.1.** A fuzzy set $\mu \in [0, 1]^R$ is a fuzzy number, provided there is a closed interval $[a, b] \in \mathbb{R}$, such that

1. $\mu(x) = 1$, for all $x \in [a, b]$.

2. It exists a function $L : (-\infty, a) \to [0, 1]$, that is continuous and non-decreasing, such that $\mu(x) = L(x)$, for all $x \in (-\infty, a)$.

3. It exists a function $R : (b, +\infty) \to [0, 1]$, that is continuous and decreasing, such that $\mu(x) = R(x)$, for all $x \in (b, +\infty)$. 
The following are examples of fuzzy numbers.

6.2.2

It can be proven that, if * ∈ {+, −, ·, ÷}, and if μ, σ are fuzzy numbers, then μ ⊗ σ is also a fuzzy number. For example, with the fuzzy number,

\[ mu_3(x) = \begin{cases} 
    x - 2, & \text{if } x \in (2, 3) \\
    4 - x, & \text{if } x \in (3, 4) \\
    0, & \text{otherwise,}
\end{cases} \]

for which \([a, b] = [3, 3] = \{3\}, L(x) = x - 2, R(x) = 4 - x\), it results

\[
(mu_3 \oplus mu_3)(t) = \sup_{t = x+y} \min(mu_3(x), mu_3(y)) = \begin{cases} 
    \frac{t - 4}{2}, & \text{if } t \in [9, 6] \\
    \frac{8 - t}{2}, & \text{if } t \in [6, 8] \\
    0, & \text{otherwise}
\end{cases}
\]

since for \(t < 4\), or \(x + y < 4\), and for \(t > 8\), or \(x + y > 8\), it should be \((mu_3 \oplus mu_3)(t) = 0\); for \(t = 3\), or \(x + y = 3\), it should be \((mu_3 \oplus mu_3)(3) = 1\), and:
6.2. FUZZY NUMBERS

- For \( t \in [4, 6] \), \( L \) is the segment joining \((4, 0)\) and \((b, 1)\), that is

\[
0 = \begin{bmatrix}
  x & y & 1 \\
  4 & 0 & 1 \\
  6 & 1 & 1
\end{bmatrix} = -x + 2y + 4, \text{ or } y = \frac{x - 4}{2}
\]

- For \( t \in [6, 8] \), \( R \) is the segment joining \((6, 1)\) and \((8, 0)\), that is

\[
0 = \begin{bmatrix}
  x & y & 1 \\
  6 & 1 & 1 \\
  8 & 0 & 1
\end{bmatrix} = x + 2y - 8, \text{ or } y = \frac{8 - x}{2}
\]

Graphically,

Notice that the result is a fuzzy number \( \mu_6 \), such that the amplitude of it \(([8, 4], 8 - 4 = 4)\) is twice than that of the initial \( \mu_3([2, 4], 4 - 2 = 2) \). When operating with fuzzy numbers, the amplitudes grow.

6.2.3

Let’s show a systematic way of operating with triangular fuzzy numbers, that is, those represented by

\[
\mu_n(x) = \begin{cases}
    L(x), & \text{if } x \in (a, n) \\
    R(x), & \text{if } x \in (b, n) \\
    0, & \text{otherwise},
\end{cases}
\]

with \( L \) and \( R \) linear functions, and \( \mu_n(n) = 1 \). Graphically,
Namely, take

\[ \mu_n(x) = \begin{cases} 
1 + \frac{x-n}{\alpha}, & \text{if } x \in (n-\alpha, n) \\
1 + \frac{n-x}{\alpha}, & \text{if } x \in (n, n+\alpha) \\
0, & \text{otherwise},
\end{cases} \]

and

\[ \mu_m(x) = \begin{cases} 
1 + \frac{x-m}{\beta}, & \text{if } x \in (m-\beta, m) \\
1 + \frac{m-x}{\beta}, & \text{if } x \in (m, m+\beta) \\
0, & \text{otherwise},
\end{cases} \]

Obviously \((\mu_n \oplus \mu_m)(n + m) = 1\); if \(t \leq m + n(\alpha + \beta)\), \((\mu_n \oplus \mu_m)(t) = 0\); if \(t \geq m + n + (\alpha + \beta)\), \((\mu_n \oplus \mu_m)(t) = 0\). Hence, the problem is reduced to compute the values of \(\mu_n \oplus \mu_m\) in the two intervals \((m + n - (\alpha + \beta), m + n)\), and \((m + n, m + n + (\alpha + \beta))\). It can be done as follows:

1. If \(x \in (n-\alpha, n)\), and \(y \in (m-\beta, m)\), \(t = x + y \in (n+m-(\alpha+\beta), n+m)\), and, since \(\mu_n(x) = 1 + \frac{x-n}{\alpha}\), \(\mu_m(y) = 1 + \frac{y-m}{\beta}\), it follows (with \(\mu_n(x) = \mu_m(y) = z\)): \(x = \alpha z + n - \alpha\), \(y = \beta z + m - \beta\), and

\[ t = x + y = (\alpha + \beta)z + n + m - \alpha - \beta, \]
6.2. FUZZY NUMBERS

6.2.4

Let’s take into account the substraction \( \mu_n \ominus \mu_m = \mu_n \ominus [(\neg 1) \ominus \mu_m] \), with 
\[
((-1) \ominus \mu_m)(t) = \mu_m\left(\frac{t}{-1}\right) = \mu_m(-t),
\]
that means

\[
\begin{align*}
\mu_n - \mu_m &= \mu_n - \left[(\neg 1) \ominus \mu_m\right] = \mu_n - \mu_m(\neg 1) \\
&= \mu_n - \mu_m(\neg 1) \\
&= \mu_n - \mu_m(\neg 1)
\end{align*}
\]

For example, \( \mu_3 \) with \( \beta = 1 \), and \( \mu_8 \) with \( \beta = 2 \), give \( \mu_3 \ominus \mu_8 = \mu_{11} \)
with \( \alpha + \beta = 3 \), that is the triangle is on \([11 - 3, 11 + 3] = [8, 14]\).
with $-\mu_m = (-1)\cap\mu_m$. The fuzzy number $-\mu_m$ is obtained by translating $\mu_m$ symmetrically at the other side of $x = 0$. For example, with

$$\mu_3(x) = \begin{cases} x - 2, & \text{if } x \in (2, 3) \\ 4 - x, & \text{if } x \in (3, 4) \\ 0, & \text{otherwise,} \end{cases}$$

it results

$$(\mu_3 \cap \mu_3)(t) = \begin{cases} \frac{t+2}{2}, & \text{if } x \in (-2, 0) \\ \frac{2-t}{2}, & \text{if } x \in (0, 2) \\ 0, & \text{otherwise.} \end{cases} = \mu_0(t), \text{ with the interval } [a, b] = [-2, +2].$$

6.2.5

For what concern the product $\mu_n \cap \mu_m$ of two triangular fuzzy numbers, the method is the same of the sum but remembering that since $(\mu_n \cap \mu_m)(t) = \sup\min_{t=x,y}(\mu_n(s), \mu_m(y))$, the value $t$ will be reached by the product $x \cdot y$. For example, with $\mu_3$ as below, let’s compute $\mu_3 \cap \mu_3$. The process is as follows.

1. If $t \leq 4$, and $x \cdot y = t$, either $x \leq 2$, or $y \leq 2$. Hence, $(\mu_3 \cap \mu_3)(t) = 0$.

Analogously, if $t \geq 16$, $(\mu_3 \cap \mu_3)(t) = 0$.

Obviously, $(\mu_3 \cap \mu_3)(9) = 1$.

2. Hence, $L$ will be defined in $[4, 9]$, and $R$ in $[9, 16]$.

3. If $x, y \in [2, 3]$, it is $x \cdot y \in [4, 9]$, hence,

$$L(t) = \sup\min_{t=x,y}(x - 2, y - 2) = \alpha \Rightarrow \alpha = x - 2, \alpha = y - 2 \Rightarrow$$

$$\Rightarrow x = \alpha + 2, y = \alpha + 2 \Rightarrow t = x \cdot y = (\alpha + 2)^2 \Rightarrow \alpha = \sqrt{t} - 2.$$

4. If $x, y \in [3, 4]$, it is $x \cdot y \in [9, 16]$, hence,

$$R(t) = \sup\min_{t=x,y}(4 - x, 4 - y) = \alpha \Rightarrow \alpha = 4 - x, \alpha = 4 - y \Rightarrow$$
\[ x = 4 - \alpha, \ y = 4 - \alpha \Rightarrow t = (4 - \alpha)^2 \Rightarrow \alpha = 4 - \sqrt{t}. \]

Finally,

\[
(\mu_3 \odot \mu_3)(t) = \begin{cases} 
\sqrt{t} - 2, & \text{if } t \in [4, 9] \\
4 - \sqrt{t}, & \text{if } t \in [9, 16] \\
0, & \text{otherwise.}
\end{cases} = \mu_9(t).
\]

Graphically

\[ \text{Remark 6.2.2. It should be pointed out that, in the case of the product of triangular fuzzy numbers, the result is not a triangular (linear) fuzzy number, since functions } L \text{ and } R \text{ are not linear. In addition the interval } [a, b] \text{ is not symmetrical. Look that for } \mu_9 \text{ is [4, 16] whose mid point is not 9 but 10.} \]

6.2.6

Let’s now consider the quotient \( \oplus \) of fuzzy numbers defined by

\[
(\mu_n \oplus \mu_m)(t) = \sup_{t=\frac{x}{y}} \min(\mu_n(x), \mu_m(y)),
\]

or \( \mu_n \oplus \mu_m = \mu_n \odot \frac{1}{\mu_m} \), with \( \frac{1}{\mu_m}(x) = \begin{cases} 
\mu_m(\frac{1}{x}), & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases} \), and \( \frac{1}{\mu_m}(\frac{1}{m}) = 1. \)
Let’s compute the example $\mu_3 \oplus \mu_3$, with $\mu_3$ as in 6.2.4. It will be:

$$(\frac{1}{\mu_3})(y) = \begin{cases} 
4 - \frac{1}{y}, & \text{if } y \in \left(\frac{1}{4}, \frac{1}{3}\right) \\
\frac{1}{y} - 2, & \text{if } y \in \left(\frac{1}{3}, \frac{1}{2}\right) \\
0, & \text{otherwise.}
\end{cases}$$

Hence, for $\mu_3 \oplus \mu_3 = \mu_3 \ominus \frac{1}{\mu_3}$, it will be $(\mu_3 \oplus \mu_3)(1) = 1$, and

- $L$, $\alpha = 4 - \frac{1}{y} = x - 2 \Rightarrow t = x \cdot y = \frac{\alpha + 2}{\alpha - 2} \Rightarrow \alpha = \frac{4t - 2}{t + 1}$
- $R$, $\alpha = 4 - x = \frac{1}{y} - 2 \Rightarrow t = x \cdot y = \frac{4 - \alpha}{\alpha + 2} \Rightarrow \alpha = \frac{4 - 2t}{t + 1}$

That is

$$(\mu_3 \oplus \mu_3)(t) = \begin{cases} 
\frac{4t - 2}{t + 1}, & \text{if } t \in \left[\frac{1}{2}, 1\right] \\
\frac{4 - 2t}{t + 1}, & \text{if } t \in [1, 2] \\
0, & \text{otherwise}
\end{cases} = \mu_1(t)$$

Graphically,

Remark 6.2.3. Obviously, like in the product’s case, the result is not a linear-triangular fuzzy number, and the basic interval $[a, b]$ is not symmetrical respect to the point $\frac{n}{m}$.
6.2. FUZZY NUMBERS

6.2.7

As it is easy to see, if \( \mu, \sigma \in [0, 1]^R \), and \( * \in \{+, -, \cdot, \} \), it follows the relation between the corresponding \( \alpha \)-cuts:

\[
(\mu \circ \sigma)_\alpha = \mu_\alpha \circ \sigma_\alpha,
\]

with the particular supposition that when \( * = \cdot \), it should be required that \( 0 \notin \mu_\alpha \) for all \( \alpha \in (0, 1] \). Notice that it also follows

\[
\mu \circ \sigma = \bigcup_{\alpha \in [0, 1]} \alpha \land (\mu \ast \sigma)_\alpha = \bigcup_{\alpha \in [0, 1]} \alpha \land [\mu_\alpha \circ \sigma_\alpha].
\]

It is for this equality that the beforehand computations were made. For example, with

\[
(\mu_1)(x) = \begin{cases} \frac{x+1}{2}, & \text{if } t \in (-1, 1) \\ \frac{3-x}{2}, & \text{if } t \in (1, 3) \\ 0, & \text{otherwise} \end{cases}, \quad (\mu_3)(x) = \begin{cases} \frac{x-1}{2}, & \text{if } t \in (1, 3] \\ \frac{5-x}{2}, & \text{if } t \in (3, 5) \\ 0, & \text{otherwise} \end{cases}
\]

the corresponding \( \alpha \)-cuts are

\[
(\mu_1)_\alpha = [2\alpha - 1, 3 - 2\alpha], \quad (\mu_3)_\alpha = [2\alpha + 1, 5 - 2\alpha],
\]

with which

\[
[\mu_1 \oplus \mu_3]_\alpha = [4\alpha, 8 - 4\alpha], \text{ for } \alpha \in (0, 1].
\]

Hence,

\[
(\mu_1 \oplus \mu_3)(t) = \begin{cases} \frac{t}{4}, & \text{if } t \in (0, 4] \\ \frac{8-t}{4}, & \text{if } t \in (4, 8] = \mu_4(t) \\ 0, & \text{otherwise} \end{cases}
\]
6.3 A note on the lattice of fuzzy numbers

6.3.1

As it is well known, \((\mathbb{R}, \min, \max)\) is a distributive lattice that come from the totally ordered set \((\mathbb{R}, \leq)\). The order \(\leq\) is definable from the lattice operations \(\min, \max\) by

\[ a \leq b \iff a = \min(a, b) \iff b = \max(a, b). \]

In addition, with \(a' = 1-a\), it is \(\min(a, b) = (\max(a', b'))'\), and \(\max(a, b) = (\min(a', b'))'\). Let’s extend these operations to the set \(\mathbb{R}^*\) of all fuzzy numbers, by

\[
(\mu \otimes \sigma)(t) = \sup_{t=\min(x,y)} (\mu \times \sigma)(x, y), \quad \text{and} \quad (\mu \boxtimes \sigma)(t) = \sup_{t=\max(x,y)} (\mu \times \sigma)(x, y).
\]

Without the proof, it holds

**Theorem 6.3.1.** \((\mathbb{R}^*, \boxtimes, \otimes)\) is a distributive lattice, that means:

- \(\mu \otimes \sigma, \mu \otimes \sigma \in \mathbb{R}^*\)
- \(\mu \otimes \sigma = \sigma \otimes \mu, \mu \otimes \mu = \mu\)
- \(\mu \otimes \sigma = \sigma \otimes \mu, \mu \otimes \mu = \mu\)
- \(\mu \otimes (\mu \otimes \sigma) = \mu, \mu \otimes (\mu \otimes \sigma) = \mu\)
- \(\mu \otimes (\sigma \otimes \lambda) = (\mu \otimes \sigma) \otimes (\mu \otimes \lambda)\)
- \(\mu \otimes (\sigma \otimes \lambda) = (\mu \otimes \sigma) \otimes (\mu \otimes \lambda)\),

for all \(\mu, \varphi, \lambda\) in \(\mathbb{R}^*\).

Hence, \(\mathbb{R}^*\) can be endowed with the partial order given by

\[ \mu \leq^* \sigma \iff \mu \otimes \sigma = \mu \iff \mu \otimes \sigma = \sigma. \]
6.3.2

With \( X = \{1, 2, 3\} \), and \( \mu = 0.8|1 + 0.7|2 + 1|3, \sigma = 0.9|1 + 1|2 + 0.6|3 \), compute \( \mu \otimes \sigma, \mu \otimes \sigma \).

Since, \( t = \min(x,y) \) and \( t = \max(x,y) \) belong to \( \{1, 2, 3\} \), it results:

- \( \mu \otimes \sigma(t) = \max_{t=\min(x,y)} [\min(\mu(x), \sigma(y))] \), and:
  \[
  \begin{align*}
  (\mu \otimes \sigma)(1) &= \max(\min(\mu(1), \sigma(1)), \min(\mu(1), \sigma(2)), \min(\mu(2), \sigma(1)), \\
                             &\quad \min(\mu(1), \sigma(3)), \min(\mu(3), \sigma(1))) = \max(\min(0.8, 0.9), \min(0.8, 1), \\
                             &\quad \min(0.7, 0.9), \min(0.8, 0.6), \min(1, 0.9)) = \max(0.8, 0.8, 0.7, 0.6, 0.9) = 0.9 \\
  (\mu \otimes \sigma)(2) &= \max(\min(\mu(2), \sigma(3)), \min(\mu(3), \sigma(2))) = \\
                         &\quad \max(\min(0.7, 0.6), \min(1, 1)) = \max(0.6, 1) = 1 \\
  (\mu \otimes \sigma)(3) &= \min(\mu(3), \sigma(3)) = \min(1, 0.6) = 0.6
  \end{align*}
\]

that is \( \mu \otimes \sigma = 0.9|1 + 1|2 + 0.6|3 \).

This fuzzy set is different from \( \mu \cdot \sigma = 0.8|1 + 0.7|2 + 0.6|3 \), with \( \cdot \) the t-norm \( \min \).

- \( \mu \otimes \sigma(t) = \max_{t=\max(x,y)} [\min(\mu(x), \sigma(y))] \), and:
  \[
  \begin{align*}
  (\mu \otimes \sigma)(1) &= \min(\mu(1), \sigma(1)) = \min(0.8, 0.9) = 0.8 \\
  (\mu \otimes \sigma)(2) &= \max(\min(\mu(1), \sigma(2)), \min(\mu(2), \sigma(1))) = \\
                         &\quad \max(\min(0.8, 1), \min(0.7, 0.9)) = \max(0.8, 0.7) = 0.8 \\
  (\mu \otimes \sigma)(3) &= \max(\min(\mu(1), \sigma(3)), \min(\mu(3), \sigma(1)), \min(\mu(2), \sigma(3)), \\
                         &\quad \min(\mu(3), \sigma(2)), \min(\mu(3), \sigma(3))) = \max(\min(0.8, 0.6), \min(1, 0.9), \\
                         &\quad \min(0.7, 0.6), \min(1, 1), \min(1, 0.6)) = 1,
  \end{align*}
\]

that is \( \mu \otimes \sigma = 0.8|1 + 0.8|2 + 1|3 \).

This fuzzy set is different from \( \mu + \sigma = 0.9|1 + 1|2 + 1|3 \), with \( + \) the t-conorm \( \max \).
**Remark 6.3.2.** It is easy to check that, although it is not \( \mu \leq \sigma \) pointwise, it is \( \mu \otimes \sigma \leq^* \mu \otimes \sigma \).

**Remark 6.3.3.** Since \( t = \min(x, y) \) means

- \( t = x \), if \( x \leq y \),
- \( t = y \), if \( y \leq x \),

it is immediate that

\[
(\mu \otimes \sigma)(t) = \max[\sup_{t \leq x} \min(\mu(x), \sigma(t)), \sup_{t \leq y} \min(\mu(t), \sigma(y))],
\]

a formula that facilitates to obtain \( \mu \otimes \sigma \), given \( \mu \) and \( \sigma \). Analogously, since \( t = \max(x, y) \) means \( t = x \), if \( y \leq x \), and \( t = y \), if \( y \geq x \), there is a similar formula for \( \mu \otimes \sigma \).

For example, in the case of \( \mu \) and \( \sigma \) given in the figure

```
1

\[
\begin{array}{c}
\mu \\
\sigma \\
\end{array}
\]

0

\[
\begin{array}{c}
a_1 \\
b_1 \\
a_2 \\
b_2 \\
\end{array}
\]
```

it results:

1. If \( t \leq a_1 \), \( (\mu \otimes \sigma)(t) = 0 \)
2. If \( a_2 \leq t \), \( (\mu \otimes \sigma)(t) = 0 \)
3. If \( a_1 \leq t \leq b_1 \), \( (\mu \otimes \sigma)(t) = \mu(t) \)
4. If \( b_1 \leq t \leq c \), \( (\mu \otimes \sigma)(t) = \sigma(t) \)
5. If \( c \leq t \leq a_2 \), \( (\mu \otimes \sigma)(t) = \mu(t) \).

Hence, \( \mu \otimes \sigma = \mu \).
6.4  A note on fuzzy quantifiers

6.4.1
In classical logic there are only considered the two quantifiers $\forall$ (for all), the universal quantifier, and $\exists$ (it exists, or for some), the existential quantifier, with the addition of $\exists!$ (it exists only one). For example, given a sequence of real numbers $(a_n)$, it is said that the real number $a$ is its limit, when

$$\forall \epsilon > 0, \exists k \in \mathbb{N}, \forall n \in \mathbb{N} : [(n > k) \rightarrow (|a_n - a| < \epsilon)].$$

Analogously, a function $f : [a, b] \rightarrow \mathbb{R}$ is bounded, when

$$\exists M \in \mathbb{R}^+, \forall x \in [a, b] : [|f(x)| < M].$$

The importance of these two quantifiers to clearly write mathematical expressions does not need to be stressed. Nevertheless, both in arithmetic computing and in natural language more quantifiers are needed and used.

Examples of arithmetical quantifiers are the percentages and the integrals. For example

- The 85% of the employees are married
- Between the 40 and the 70 percent of the employees are single.

For example, if it is known that

- The 35% of the employees are married
- The 25% of the married employees are young

What can be said on the employees that are young?

A question which answer is, obviously, $35 \times 25 = 875$, that is, at least the 8.75% of the employees are young.

Another example is
Between 15 and 25 employees are married
Between 5 and 10 married employees are young
What can be said on the employees that are young?

in which what matters is the length of the two intervals \([15, 25]\) \((l = 10)\), and \([5, 10]\) \((l = 5)\), that give at least, the interval \([5, 10 + (10 - 5)] = [5, 15]\) of employees that are young.

\[6.4.2\]

In natural language imprecise quantifiers like ‘about five’, ‘about half’, ‘most’, etc., appear and can be represented by means of fuzzy numbers. For example,

\[\text{These fuzzy quantifiers are of two main types}\]

\[\text{• } \textit{Absolute quantifiers}, \text{ when are fuzzy numbers in } \mathbb{R} \text{ (independent of the cardinality of the universe of discourse)}\]

\[\text{• } \textit{Relative quantifiers}, \text{ when are fuzzy numbers in } [0, 1] \text{ (dependent of the cardinality of the universe of discourse)}\]

Of course, the “interval” arithmetical quantifiers belong to the class of absolute (fuzzy) quantifiers as a (crisp) particular case, and the “percentage” arithmetical quantifiers belong to the class of relative (fuzzy) quantifiers as a (crisp) particular case.

The important problem is to compute the degree of validity, or truth, of statements with fuzzy quantifiers, like for example:
6.4. A NOTE ON FUZZY QUANTIFIERS

- In a given class, there are about three students whose fluency in English is low, that firstly should be represented in fuzzy terms. To do that, call \( \mu_{A_3} \) the fuzzy quantifiers, and \( \mu_{LF} \) the degree of low English-fluency, and rewrite the above statement as

- There are \( \mu_{A_3}, \mu_{LF} \)'s.

Then, the degree of validity can be taken as

\[
t = \mu_{A_3}(|\mu_{LF}|), \text{ with } |\mu_{LF}| \text{ the cardinality of } \mu_{LF}.
\]

For example, with

and the scores of \( LF \) of the (supposedly) five students \( \{1, 2, 3, 4, 5\} \) in the class, given by

\[
\mu_{LF} = 0|1 + 0|2 + 0.75|3 + 1|4 + 0.5|5,
\]

is is \( |\mu_L| = 1 + 0.75 + 0.5 = 2.25 \), and

\[
t = \mu_{A_3}(2.25) = 2.25 - 2 = 0.25,
\]

since the equation of the line joining the points \((2, 0)\) and \((3, 1)\), in the figure of \( \mu_{A_3} \), is \( y = x - 2 \).
Another, more general, kind of quantified fuzzy statements, is

There are \( Q \in X \), such that \( "F_1(x) \text{ is } P_1", \ldots, \text{ "} F_n(x) \text{ is } P_n \text{"} \),

with \( X \) the universe of discourse, \( F_i : X \to F_i(X) \subset \mathbb{R} \), \( 1 \leq i \leq n \), and \( P_i \) a predicate in \( F_i(X) \), \( 1 \leq i \leq n \). For example,

There are about 6 employees in the company that are young and whose computer skills are high.

where \( X = \{x_1, \ldots, x_n\} \) is the set of employees, \( Q = \text{about 6} \), \( F_1 = \text{computer skills} \), and \( F_2 = \text{high} \). These statement can be compressed to the form

There are \( Q H_1 \text{'s } H_2 \text{'s } \equiv \text{ There are } Q (H_1 \text{ and } H_2) \),

with \( H_1(x) = \mu_{P_1}(H_1(x)) \), \( H_2(x) = \mu_{P_2}(H_2(x)) \), for all \( x \in X \), and that correspond with the rewriting.

There are about 6 employees that are young and with high computer skills, of the given statement. Finally, with \( Z = |H_1 \cap H_2| = \sum_{i=1}^{n} min(\mu_{P_1}(F_1(x_i)), \mu_{P_2}(F_2(x_i))) \), and \( Q(Z) = Q(|H_1 \cap H_2|) \), results the more compressed form

\[ Z \text{ is } Q \]

corresponding to

The number of employees that are young and with high computer skills, is about 6,

and that gives the truth-value \( t = Q(Z) \)

The compression of fuzzy quantified statements is essential for its representation and for computing its truth-value.
Let’s consider the case with relative quantifiers. The most simple case is

- Among the $x \in X$, $Q$ are such that “$F(x)$ is $P$”.

for example,

- Among the company’s employees almost all are young.

This statement can be compressed to

- $Q$ are $H$’s, with $H(x) = \mu_P(F(x))$,

  corresponding to

- Almost all company’s employees are young.

Finally, the last compression can be done in the form:

$$Z \text{ is } Q,$$

with $Z = \frac{|H|}{|X|}$, that gives the truth value $t = Q(Z)$.

**Example 6.4.1. A problem of inference**

Given $n$ statements of the compressed form “$Z_i$ is $Q_i$”, $1 \leq i \leq n$ with $Q_i$ absolute or relative quantifiers, which statement of the form “$Z_i$ is $Q$” can be inferred? For example,

- There are about 10 workers in the establishment
- About half of the establishment workers are women

We search $Q$ such that: There are $Q$ women in the establishment

Now, it is $Q_1 =$about 10, $Q_2 =$about half, $X_1 =$set of workers, $X_2 =$set of women $\subset X_1$. Then the syllogism can be stated by

$$Z_1 \text{ is } Q_1$$
$$Z_2 \text{ is } Q_2$$
where $Z_1 = |X_1|$, $Z_2 = \frac{|X_1 \cup X_2|}{|X_1|} = \frac{|X_2|}{|X_1|}$. Hence the conclusion is “$Z$ is $Q$”, with $Z = |X_2|$, and it rests to compute $Q$.

A rule often used to obtain $Q$ is the following. If there exists $f : \mathbb{R}^n \to \mathbb{R}$ such that $Z = f(Z_1, \ldots, Z_n)$, take $Q = f(Q_1, \ldots, Q_n)$. In the example, it is $Z_1 \cdot Z_2 = |X_2| = Z$, hence $Q = Q_1 \cdot Q_2$, and the inference is

$$Z_1 \cdot Z_2 \text{ is } Q_1 \cdot Q_2, \text{ with } \mu_{Q_1} \cdot \mu_{Q_2} = \min(\mu_{Q_1}, \mu_{Q_2}).$$

As a final example,

Most of the workers are young
About half of the young workers are women

Most $\times$ About half workers are young women.
Chapter 7

Fuzzy measures

7.1 Introduction

Fuzzy sets not only appear by representing imprecise predicates, but also by representing partial or incomplete information. This is the case, for example, of a function $X$ taking values in $[0, 10]$, of which it is only known that

$$5 \leq X \leq 7, \ 8 \nleq X, \text{ and } X \nleq 2.$$ 

This information can be represented by means of a fuzzy set $\mu_X \in [0, 10]^{\Omega}$, by taking:

- $\mu_X(x) = 1 \iff X$ takes the value $x$
- $\mu_X(x) = 0 \iff X$ don’t takes the value $x$
- $\mu_X(x) = a \cdot x + b \iff$ It is unknown if $X$ can take the value $x$.

In the case of the above information:

$$\mu_X(x) = \begin{cases} 0 & \text{, if } x \in [0, 2] \cup [8, 10] \\ \frac{x-2}{3} & \text{, if } x \in [2, 5] \\ 1 & \text{, if } x \in [5, 7] \\ 8-x & \text{, if } x \in [7, 8] \end{cases}$$
In these cases, the questions of the uncertainty concerning the questions,

- $X \in A$ for $A \subset [0, 100]$
- $X \in A$, for $\mu_A \in [0, 10]^X$,

arise and suggests the problem of its measuring. In this section, several “measures” of that king of uncertainty, including probabilities when applicable, are introduced and studied.

The area of a polygon is a measure of its extensional size, and the length of a segment is a measure of its longitudinal size. Everybody knows several examples of measures, as it is, for example, the number of apples in a basket. But, how can it be the concept of measure formalized? We will consider the following kind of measures.

### 7.2 The concept of a measure

Given a ground set $X$, and provided

- $\mathcal{F} \subseteq [0, 1]^X$ is a family of fuzzy subsets of $X$, such that $\mu_0, \mu_1 \in \mathcal{F}$, and that $\preceq$ is a preorder in $\mathcal{F}$ translating a qualitative relation between the elements in $\mathcal{F}$,

- $(L, \preceq)$ is a preordered set with first element 0,

we will say that $m : \mathcal{F} \to L$ is a $(L, \preceq)$-measure, whenever

1. $m(\mu_0) = 0$
2. If $\mu \preceq \sigma$, then $m(\mu) \preceq m(\sigma)$.

**Example 7.2.1.** Take $\mathcal{F} = [0, 1]^X$, $(L, \preceq) = ([0, 1], \leq)$ the unit interval with the partial linear order $\leq$ of the real line, and the qualitative relation “$\mu$ is less fuzzy than $\sigma$”, translated by the so-called sharpened order $\leq_S \,(=\leq)$ defined by
7.2. THE CONCEPT OF A MEASURE

\[ \mu \preceq_S \sigma \iff \begin{cases} \mu(x) \leq \sigma(y), & \text{if } \sigma(x) \leq 1/2 \\ \sigma(x) \leq \mu(x), & \text{if } \sigma(x) > 1/2, \end{cases} \]

that is a reflexive, transitive (and antisymmetric), crisp relation. The fuzzy set \( \mu_{0.5} \) is the highest one and all crisp sets \( \mu \in \{0,1\}^X \) are minimals in \((\mathfrak{F}, \preceq_S)\). Any mapping, \( m : [0,1]^X \to [0,1] \) such that

- If \( \mu \) is crisp, then \( m(\mu) = 0 \)
- \( m(\mu_{0.5}) = 1 \)
- If \( \mu \preceq_S \sigma \), then \( m(\mu) \leq m(\sigma) \),

is a \(([0,1], \preceq)\)-measure since \( m(\mu_0) = 0 \) because of \( \mu_0 \in \{0,1\}^X \). These measures are called measures of fuzziness, of fuzzy entropies.

If \( X = \{x_1, \ldots, x_n\} \) is finite, the following mappings are examples of fuzzy entropies:

1. \( m(\mu) = 1 - 2 \max_{1 \leq i \leq n} |\mu(x_i) - \frac{1}{2}| \)
2. \( m(\mu) = 2 \max_{1 \leq i \leq n} \mu(x_i) \cdot (1 - \mu(x_i)) \)
3. \( m(\mu) = \sum_{i=1}^n \sigma(\mu(x_i)), \text{ with } \sigma(x) = x \ln x - (1-x) \ln(1-x) \) (logarithmic entropy).
4. \( m(\mu) = \frac{1}{2n} \sum_{i=1}^n |\mu(x_i) - \mu_{C_\mu}(x_i)|, \text{ with } \mu_{C_\mu}(x) = \begin{cases} 1, & \text{if } \mu(x) > 0.5 \\ 0, & \text{if } \mu(x) \leq 0.5 \end{cases} \) the closest crisp set to \( \mu \) (linear index of fuzziness)
5. \( m(\mu) = \frac{1}{2n} \sqrt{\sum_{i=1}^n (\mu(x_i) - \mu_{C_\mu}(x_i))^2} \) (quadratic index of fuzziness).

**Remark 7.2.2.** 1. For some specific problems, measures of fuzziness are selected verifying the additional property of symmetry:

- For some negation \( N \), it is \( m(\mu) = m(N \circ \mu) = m(\mu') \).
For example, with $N = 1-id$, measures 1, 2, 3, 4 do verify this property of symmetry $m(\mu) = m(1 - \mu)$.

2. With each measure of fuzziness $m$, it can be defined a measure of bodianity $1 - m$, that, obviously verifies:

- the value of $1 - m$ is 1 for the crisp sets.
- the value of $1 - m$ is 0 for $\mu_{1/2}$.
- $1 - m$ is non-increasing with respect to the order $\leq_S$.

**Example 7.2.3.** Take $\mathcal{F} \in [0, 1]^X$, with the partial pointwise order `$\mu \leq \sigma$ if and only if $\mu(x) \leq \sigma(x)$, for all $x \in X$' (and such that $\mu_0, \mu_1 \in \mathcal{F}$). A mapping $m : [0, 1]^X \to [0, 1]$ is a fuzzy measure provided $m$ verifies:

1. $m(\mu_0) = 0$
2. $m(\mu_1) = 1$
3. If $\mu \leq \sigma$, then $m(\mu) \leq m(\sigma)$.

When $\mathcal{F} \in \{0, 1\}^X \approx \mathcal{P}(X)$, a fuzzy measure should verify

1. $m(\emptyset) = 0$
2. $m(X) = 1$
3. If $A \subseteq B$, then $m(A) \subseteq m(B)$.

For example, if $X$ is a finite set $\{x_1, \ldots, x_n\}$, and $|\mu| = \sum_{x_i \in X} \mu(x_i)$, the crisp cardinality of $\mu$, then the function $m(\mu) = \frac{|\mu|}{n}$, is a fuzzy measure.

Notice, that since $\mu \cdot \sigma = T \circ (\mu \times \sigma)$, $\mu + \sigma = S \circ (\mu \times \sigma)$, it is

$$\mu \cdot \sigma \leq \mu, \mu \cdot \sigma \leq \sigma, \mu \leq \mu + \sigma, \sigma \leq \mu + \sigma,$$

provided $\mu \cdot \sigma, \mu + \sigma \in \mathcal{F}$. Then, for all fuzzy measure $m$, is:

$$m(\mu \cdot \sigma) \leq m(\mu), m(\mu \cdot \sigma) \leq m(\sigma), m(\mu) \leq m(\mu + \sigma), m(\sigma) \leq m(\mu + \sigma),$$
7.3. TYPES OF MEASURES

and

\[ m(\mu \cdot \sigma) \leq \min(m(\mu), m(\sigma)) \leq \max(m(\mu), m(\sigma)) \leq m(\mu + \sigma). \]

In the particular case in which \( \mu, \sigma \in \{0, 1\}^X \), it results

\[ m(A \cap B) \leq \min(m(A), m(B)) \leq \max(m(A), m(B)) \leq m(A \cup B), \]

for all \( A, B \in \mathcal{P}(X) \). Notice that fuzzy measures are applied to both \([0, 1]^X\) and \(\mathcal{P}(X)\).

7.3 Types of measures

Given a triplet \((X, \mathcal{F}, m)\), where \(m\) is a fuzzy measure, if for some negation \('N) and some union \('S), is

1. When \( \mu \leq \sigma' \), then \( m(\mu + \sigma) \leq m(\mu) + m(\sigma) \), \(m\) is sub-additive

2. When \( \mu \leq \sigma' \), then \( m(\mu + \sigma) \geq m(\mu) + m(\sigma) \), \(m\) is super-additive,

and when \(m\) is both sub-additive and super-additive, that is

\[ m(A \cap B) = m(A) + m(B), \] \(m\) is additive.

This classification (once completed with those measures that are neither sub-additive, nor super-additive), in the case in which \( \mu, \sigma \in \{0, 1\}^X \), is

- If \( A \cap B = \emptyset \), and \( m(A \cup B) \leq m(A) + m(B) \), \(m\) is sub-additive

- If \( A \cap B = \emptyset \), and \( m(A \cup B) \geq m(A) + m(B) \), \(m\) is super-additive

- If \( A \cap B = \emptyset \), and \( m(A \cup B) = m(A) + m(B) \), \(m\) is additive.

Example 7.3.1. The measure \( m(A) = \frac{|A|}{n} \), in a finite set \( X = \{x_1, \ldots, x_n\} \), is additive.
\section{\lambda-measures}

With $\mathcal{F} = \mathcal{P}(X)$, $m_\lambda : \mathcal{P}(X) \to [0, 1]$ is called a Sugeno’s $\lambda$-measure if, with $\lambda > -1$, it is:

1. $m_\lambda(\emptyset) = 0$
2. $m_\lambda(X) = 1$
3. If $A \cap B = \emptyset$, $m_\lambda(A \cup B) = m_\lambda(A) + m_\lambda(B) + \lambda m_\lambda(A)m_\lambda(B)$.

\textbf{Theorem 7.4.1.} All mapping $m_\lambda$ is, actually, a fuzzy measure.

\textit{Proof.} What lacks to be proven is that $A \subset B$ implies $m_\lambda(A) \leq m_\lambda(B)$. Since $A \subset B \Leftrightarrow B = A \cup (A^C \cap B)$, with $A \cap (A^C \cap B) = \emptyset$, it follows $m_\lambda(B) = m_\lambda(A^C \cap B) + \lambda m_\lambda(A)m_\lambda(A^C \cap B) = m_\lambda(A)[1 + \lambda m_\lambda(A^C \cap B)] + m_\lambda(A^C \cap B)$. From $\lambda > -1$, it follows $1 + \lambda m_\lambda(A^C \cap B) > 1 - m_\lambda(A^C \cap B)$, and $m_\lambda(B) > m_\lambda(A)[1 - m_\lambda(A^C \cap B)] = m_\lambda(A) - m_\lambda(A)m_\lambda(A^C \cap B) > m_\lambda(A)$. \hfill \Box

\textbf{Theorem 7.4.2.} $m_\lambda(A^C) = \frac{1 - m_\lambda(A)}{1 + \lambda m_\lambda(A)}$, for all $A \in \mathcal{P}(X)$.

\textit{Proof.} From $A \cap A^C = \emptyset$, follows $m_\lambda(X) = 1 = m_\lambda(A \cup A^C) = m_\lambda(A) + m_\lambda(A^C) + \lambda m_\lambda(A)m_\lambda(A^C) = m_\lambda(A^C)[1 + \lambda m_\lambda(A)] + m_\lambda(A)$. \hfill \Box

Notice, that it is $m_\lambda(A^C) = N_\lambda(m_\lambda(A))$, with the Sugeno’s negation $N_\lambda(x) = \frac{1 - x}{1 + \lambda x} (\lambda > -1)$.

\textbf{Remarks 7.4.3.} \begin{itemize}
\item With the t-conorm $S_\lambda(x, y) = x + y + \lambda xy$, it follows $m_\lambda(A \cup B) = S_\lambda(m_\lambda(A), m_\lambda(B))$, if $A \cap B = \emptyset$.
\item When $\lambda = 0$, $m_0$ is just a probability defined in $\mathcal{P}(X)$ since it results $m_\lambda(A \cup B) = m_\lambda(A) + m_\lambda(B)$ when $A \cap B = \emptyset$, that is, $m_0$ is an additive measure. In addition, since $N_0(x) = 1 - x$, it is $m_0(A^C) = 1 - m_0(A)$.
\item Notice that the axioms required for a $\lambda$-measure do not individuate a single one of them. For example, with $\lambda = 0$ what is obtained in the set of all probabilities on $\mathcal{P}(X)$.
\end{itemize}
7.4. \( \lambda \)-MEASURES

- If \( \lambda \in (-1, 0) \), it results

\[
A \cap B = \emptyset : m_\lambda(A \cup B) \leq m_\lambda(A) + m_\lambda(B),
\]

that is, if \( \lambda \in (0, +\infty) \), all the corresponding \( \lambda \)-measures are sub-additive. As it is easy to prove, if \( \lambda \in (0, +\infty) \), \( m_\lambda \) is super-additive.

- As it is well known, if \( X \) is a finite set \( X = \{x_1, \ldots, x_n\} \), all probabilities \( m_0 : \mathbb{P}(X) \to [0, 1] \) are defined by choosing \( n \) numbers \( m_0(\{x_i\}) \in [0, 1], 1 \leq i \leq n \), verifying \( \sum_{i=1}^n m_0(\{x_i\}) = 1 \), because

\[
1 = m_0(\{x_1, \ldots, x_n\}) = m_0(x_1) + \ldots + m_0(x_n).
\]

Something analogous happens with \( \lambda \)-measures when \( X = \{x_1, \ldots, x_n\} \). For example, if \( X = \{x_1, x_2, x_3\} \), it follows

\[
1 = m_\lambda(X) = m_\lambda(\{x_1, x_2, x_3\}) = m_\lambda(\{x_1, x_2\} \cup \{x_3\}) = m_\lambda(\{x_1, x_2\}) + m_\lambda(x_3) + \lambda m_\lambda(\{x_1, x_2\}) m_\lambda(x_3) = \\
= \sum_{i=1}^3 m_\lambda(x_i) + \lambda \sum_{1 \leq i < j \leq 3} m_\lambda(x_i) m_\lambda(x_j) + \lambda^2 m_\lambda(x_1) m_\lambda(x_2) m_\lambda(x_3).
\]

and for each \( \lambda \in (-1, +\infty) \), the values \( m_\lambda(x_i), 1 \leq i \leq 3 \), are to be taken verifying this equation that, of course, with \( \lambda = 0 \) reduces to

\[
1 = \sum_{i=1}^3 m_0(x_i).
\]

In the case that, for example, is \( m(x_1) = 0 \), follows

\[
1 = m_\lambda(x_2) + m_\lambda(x_3) + \lambda[m_\lambda(x_2) m_\lambda(x_3)] = m_\lambda(x_2) + m_\lambda(x_3) + \lambda m_\lambda(x_2) m_\lambda(x_3)
\]

that is: \( m_\lambda(x_3) = \frac{1 - m_\lambda(x_2)}{1 + \lambda m_\lambda(x_2)} \). With \( m_\lambda(x_2) = 0.7 \), results \( m_\lambda(x_3) = \frac{0.3}{1 + 0.3 \lambda} \). With \( \lambda = 1 \): \( m_1(x_3) = \frac{0.3}{1.03} = 0.29 \). That is: a measure \( m_1 \) is defined in \( X = \{x_1, x_2, x_3\} \), by \( m_1(x_1) = 0 \), \( m_1(x_2) = 0.7 \), \( m_1(x_3) = 0.29 \). Notice that, since \( m_1 \) is not a probability, it is \( \sum_{i=1}^3 m(x_i) = 0.99 < 1 \).
7.5 Measures of possibility and necessity

7.5.1

Let it be $\mathcal{F} \subset \mathcal{P}(X)$ a boolean algebra of subsets of $X$. A mapping $\pi : \mathcal{F} \rightarrow [0, 1]$ is called a measure of possibility, provided:

- $\pi(\emptyset) = \emptyset$
- $\pi(X) = 1$
- $\pi(A \cup B) = \max(\pi(A), \pi(B))$, for all $A, B$ in $\mathcal{F}$.

Notice that (3) does not require $A \cap B = \emptyset$. Actually, any of these mappings are fuzzy measures, since:

$$A \subset B \iff A \cup B = B: \pi(B) = \max(\pi(A), \pi(B)) \geq \pi(A), \text{ or } \pi(A) \leq \pi(B).$$

From $\max(\pi(A), \pi(B)) \leq \pi(A) + \pi(B)$, it follows $\pi(A \cup B) \leq \pi(A) + \pi(B)$ even if $A \cap B = \emptyset$. Hence, possibility measures are sub-additive. From $A \cup A^c = X$, it is $1 = \max(\pi(A), \pi(A^c)) \leq \pi(A) + \pi(A^c)$, or $1 - \pi(A) \leq \pi(A^c)$.

Obviously,

$$\pi(A_1 \cup A_2 \cup \ldots \cup A_n) = \max(\pi(A_1), \pi(A_2), \ldots, \pi(A_n)),$$

for all $A_1, \ldots, A_n$ in $\mathcal{F}$.

Hence, if $X = \{x_1, \ldots, x_n\}$ is a finite set, to have a possibility measure $\pi$, its values $\pi(x_i)$ do verify:

$$1 = \pi(X) = \max(\pi(x_1), \pi(x_2), \ldots, \pi(x_n)),$$

forcing that some of the values $\pi(x_i)$ should equal 1. For example, if $X = \{x_1, x_2, x_3\}$, the three values $\pi(x_1) = 0$, $\pi(x_2) = 0.5$, $\pi(x_3) = 1$, define a particular measure of possibility on $\mathcal{P}(X)$. It is, for example, $\pi(\{x_1, x_2\}) = \max(0, 0.5) = 0$, $\pi(\{x_1, x_3\}) = \max(0, 1) = 1$, etc.
Theorem 7.5.1. For each $\mu \in [0, 1]^X$ such that $\text{Sup} \mu = 1$, the mapping $
abla : A \rightarrow [0, 1]$ defined by

$$
\pi_\mu(A) = \text{Sup} \min(x, \mu_A(x)), A \in A,
$$

is a possibility measure.

Proof. $\pi_\mu(\emptyset) = \text{Sup} \min(x, 0) = 0$. $\pi_\mu(X) = \text{Sup} \min(x, 1) = \text{Sup} \mu(x) = 1$. Finally, since $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ for all $x \in X$:

$$
\pi_\mu(A \cup B) = \text{Sup} \min(x, \mu_{A \cup B}(x)) = \text{Sup} \min(x, \max(\mu_A(x), \mu_B(x))) =
$$

$$
= \text{Sup} \max(\min(x, \mu_A(x)), \min(x, \mu_B(x))) =
$$

$$
= \max(\text{Sup} \min(x, \mu_A(x)), \text{Sup} \min(x, \mu_B(x))) = \max(\pi_\mu(A), \pi_\mu(B)).
$$

It is said that $\mu$ is the possibility distribution of $\pi_\mu$, a possibility measure that can be considered as the one “conditioned” by $\mu$.

Each fuzzy set $\mu \in [0, 1]^X$ such that $\text{Sup} \mu = 1$ ($\mu$ is never self-contradictory) induces the possibility measure $\pi_\mu$. Provided $X = \{x_1, \ldots, x_n\}$ is a finite set, it results

$$
\pi_\mu(A) = \text{Max} \min(\mu(x_i), \mu_A(x_i)), \quad 1 \leq i \leq n.
$$

Hence, all fuzzy sets $\mu$ such that $\text{Sup} \mu = 1$, can be viewed as possibility distributions.

Examples 7.5.2. 1. If $X = \{x_1, x_2, x_3\}$ and $\mu = 1|x_1 + 0.2|x_2 + 0.8|x_3$, follows $\pi_\mu(X) = \text{Max}(\min(1, \mu_A(x_1)), \min(0.6, \mu_A(x_2)), \min(0.8, \mu_A(x_3)))$ that, with $A = \{x_2, x_3\}$ gives $\pi_\mu(A) = \max(0.6, 0.8) = 0.8$, and with $A = \{x_1, x_3\}$ gives $\pi_\mu(A) = \max(\min(1, 1), \min(0.6, 0), \min(0.8, 1)) = \max(1, 0, 0.8) = 1$. 
2. If $X = [0, 10], A = [5, 8]$, with $\mu$ in the next figure, it is $\pi\mu([5, 8]) = \sup_{x \in [0, 10]} \min(\mu(x), \mu([5, 18])(x)) = \sup_{x \in [0, 10]} \left\{ \begin{array}{ll} 0 & , x \in [0, 5] \cup [8, 10] \\ \mu(x) & , x \in [5, 18] \end{array} \right\}$. 

![Diagram](image)

### 7.5.2

A mapping $N : \mathcal{F} \rightarrow [0, 1]$ is a **measure of necessity** provided

- $N(\emptyset) = 0$
- $N(X) = 1$
- $N(A \cap B) = \min(N(A), N(B))$, for all $A, B$ in $\mathcal{F}$.

Since $A \subset B \iff A \cap B = A$, it is $N(A) = \min(N(A), N(B)) \leq N(B)$. Hence, all these mapping are actually fuzzy measures. From $\min(N(A), N(B)) \leq \pi(A) + \pi(B)$, it follows $N(A \cap B) \leq \pi(A) + \pi(B)$. Then $0 = N(\emptyset) = N(A \cap A^C) = \min(N(A), N(A^C))$, that implies:

Either $N(A) = 0$, or $N(A^C) = 0$, and $N(A) + N(A^C) \leq 1$, or $N(A^C) \leq 1 - N(A)$.

Obviously, if $A_1, A_2, \ldots, A_n \in \mathcal{F} : N(A_1 \cap A_2 \cap \ldots \cap A_n) = \min(N(A_1), N(A_2), \ldots, N(A_n))$. Then if $X = \{x_1, \ldots, x_n\}$, to define a necessity measure it is needed to take $N(x_1), N(x_2), \ldots, N(x_3)$ such that:
7.5. MEASURES OF POSSIBILITY AND NECESSITY

0 = N(\{x_1\} \cap \ldots \cap \{x_n\}) = \min(N(x_1), N(x_2), \ldots, N(x_n)), an equality that forces some of the values N(x_i) to be 0.

**Theorem 7.5.3.** Given a possibility measure \( \pi : \mathcal{F} \to [0, 1] \), the function \( N_\pi(A) = 1 - \pi(A^C) \), for all \( A \in \mathcal{F} \), is a necessity measure.

**Proof.**
\[
N_\pi(\emptyset) = 1 - \emptyset(X) = 0. \quad N_\pi(X) = 1 - \pi(\emptyset) = 1. \quad N_\pi(A \cap B) = 1 - \pi(A^C \cup B^C) = 1 - \max(\pi(A^C), \pi(B^C)) = \min(1 - \pi(A^C), 1 - \pi(B^C)) = \min(N_\pi(A), N_\pi(B)).
\]

**Theorem 7.5.4.** Given a necessity measure \( N : \mathcal{F} \to [0, 1] \), the function \( \pi_N(A) = 1 - N(A^C) \), for all \( A \in \mathcal{F} \) is a possibility measure.

**Proof.**
\[
\pi_N(\emptyset) = 1 - N(X) = 0. \quad \pi_N(X) = 1 - N(\emptyset) = 1. \quad \pi_N(A \cup B) = 1 - N(A^C \cap B^C) = 1 - \min(N(A^C), N(B^C)) = \max(\pi_N(A), \pi_N(B)).
\]

The pairs \((\pi, N_\pi)\) and \((N, \pi_N)\) are called dual-pairs of possibility/necessity measures. Notice that with them it can be read:

- Necessity of \( A \) = “not the possibility of not \( A \)”
- Possibility of \( A \) = “not the necessity of not \( A \)”

**Remark 7.5.5.** If \( \pi = \pi_\mu \), the corresponding \( N_{\pi_\mu} \) is given by

\[
N_{\pi_\mu}(A) = 1 - \pi_\mu(A) = 1 - \sup_{x \in X} \min(\mu(x), \mu_A(x)) = \inf_{x \in X} \max(1 - \mu(x), \mu_A(x)),
\]

that, in the finite case \( X = \{x_1, \ldots, x_n\} \), is

\[
N_{\pi_\mu}(A) = \min_{1 \leq i \leq n} \max(1 - \mu(x_i), \mu_A(x_i)).
\]

**Theorem 7.5.6.** For all dual pair \((\pi, N)\) is:

1. If \( N(A) > 0 \), then \( \pi(A) = 1 \)
2. If \( \pi(A) < 1 \), then \( N(A) = 0 \)
Proof.

\[ 1 = \max(\pi(A), \pi(A^C)), \quad \text{and} \quad 0 = \min(N(A), N(A^C)), \]

follows that if \( N(A) > 0 \), then \( N(A^C) = 0 \), and \( \pi(A) = 1 - N(A^C) = 1 \). If \( \pi(A) < 1 \), then \( \pi(A^C) = 1 \) and \( N(A) = 1 - \pi(A^C) = 0 \). \( \square \)

Remark 7.5.7. Although the proof will not be presented, let’s show the following important notice. In the case \( X \) is finite, for any possibility measure \( \pi \) it exists a (non unique!) fuzzy set \( \mu \in [0,1]^X \) with \( \sup \mu = 1 \) such that \( \pi = \pi_\mu \). In the finite case, all possibility measures come from possibility distributions.

### 7.6 Examples

**Example 7.6.1.** On the age of a person \( P \) it is only available the incomplete information given by

1. \( 37 \leq Age \leq 41 \)

2. It is neither \( Age(p) \leq 32 \), nor \( Age(p) \geq 43 \).

What can be said about the possibility and the necessity of “\( Age(p) \geq 42 \)”,” \( Age(p) \leq 40 \)”, and “\( Age(p) \geq 33 \)”?

**Solution**

the available incomplete information can be represented by the fuzzy set or possibility distribution \( \mu \):
7.6. EXAMPLES

Hence,

- \( \pi_\mu(Age(p) \geq 42) = \pi_\mu([42, 100]) = \sup_{x \in [0, 100]} \min(\mu(x), \mu_{[42, 100]}(x)) = \sup_{x \in [42, 100]} \mu(x) = \mu(42) \in (0, 1) \). Hence \( N_{\pi_\mu}(Age(p) \geq 42) = 0 \).

The value \( \mu(42) \) can be computed as follows. The segment between \( (41, 1) \) and \( (43, 0) \), verifies

\[
0 = \begin{vmatrix} x & y & 1 \\ 41 & 1 & 1 \\ 43 & 0 & 1 \end{vmatrix} = x + 2y - 43 \Rightarrow y = \frac{43 - x}{2},
\]

hence, \( \mu(42) = \frac{43 - 42}{2} = 0.5 \). The possibility of “\( Age(p) \geq 42 \)” is 0.5, and its necessity is 0.

- \( \pi_\mu(Age(p) \leq 40) = \pi_\mu([0, 40]) = \mu(40) = 1 \), and \( N_{\pi_\mu}(Age(p) \leq 40) = N_{\pi_\mu}([0, 40]) = 1 - \pi_\mu([40, 100]) = 1 - \sup_{x \in (40, 100]} \mu(x) = 1 - \mu(40) = 0 \).

The possibility of “\( Age(p) \geq 42 \)” is 1, and its necessity is 0.

- \( \pi_\mu(Age(p) \geq 33) = \pi_\mu([33, 100]) = 1 \), and \( N_{\pi_\mu}(Age(p) \geq 33) = N_{\pi_\mu}([33, 100]) = 1 - \pi_\mu([0, 33]) = 1 - \mu(33) \). This value can be computed by:

\[
0 = \begin{vmatrix} x & y & 1 \\ 32 & 0 & 1 \\ 37 & 1 & 1 \end{vmatrix} = -x + 5y + 32 \Rightarrow y = \frac{x - 32}{5}, \text{ and } \mu(33) = \frac{1}{5}.
\]

That is, \( N_{\pi_\mu}(Age(p) \geq 33) = 1 - \frac{1}{5} = \frac{4}{5} \). The possibility of “\( Age(p) \geq 33 \)” is 1, and its necessity is 0.8.

**Example 7.6.2.** It is only known that a function \( F : X \to [0, 1] \) doesn’t take any value below 1/4 but takes values above 3/4. What can be said on the imprecise statements “\( F \) is small”, and “\( F \) is not small”.

**Solution**

The available but incomplete information can be translated by the possibility distribution \( \mu \)
Let’s take $\mu_{\text{small}}(x) = 1 - x$, $\mu_{\text{not small}}(x) = x$. It is:

- $\pi_\mu(\text{F is small}) = \sup_{x \in [0,1]} \min(\mu(x), \mu_{\text{small}}(x)) = \sup_{x \in [0,1]} \min(\mu(x), 1-x) = 1/2 \Rightarrow N_{\pi_\mu}(\text{F is small}) = 1 - \pi_\mu(\text{F is not small})$

- $\pi_\mu(\text{F is not small}) = \sup_{x \in [0,1]} \min(\mu(x), \mu_{\text{not small}}(x)) = \sup_{x \in [0,1]} \min(\mu(x), x) \Rightarrow N_{\pi_\mu}(\text{F is small}) = 1 - \pi_\mu(\text{F is small})$.

Hence:

- $\pi_\mu(\text{F is small}) = 0.5$, and $N_{\pi_\mu}(\text{F is small}) = 1 - 1 = 0$.
- $\pi_\mu(\text{F is not small}) = 1$, and $N_{\pi_\mu}(\text{F is not small}) = 1 - 0.5 = 0.5$.

**Remark 7.6.3.** Notice that this example is, like the following, not with questions related to precise or crisp sets, but to impreciseness (fuzzy sets). Although Possibility Theory is introduced with crisp sets, it is also applicable to fuzzy sets within the theory given by the triplet $(\text{min}, \text{max}, 1 - \text{id})$.

**Example 7.6.4.** John is a member of a community where the predicate $Y = \text{young}$ is used following
Find the possibility and the necessity of the statement “John is around 35 years old”.

Solution. The graphics of $\mu_Y$ and $\mu_P$, with $P = Around 35$, are

Hence, $\pi_\mu(John is around 35 years old) = \sup_{x \in [0,100]} \min(\mu_Y(x), \mu_P(x)) = \frac{2}{3}$, since it does correspond with the intersection of $\mu_Y$ and $\mu_P$, that is, of the straight lines respectively given by $y = \frac{40-x}{10}$, $y = \frac{x-35}{5}$. It results $x = \frac{100}{3}$ and $y = \frac{2}{3}$. Since $\pi_\mu(John is around 35 years old) < 1$, it results $N_{\pi_\mu}(John is around 35 years old) = 0$.

7.7 Probability, possibility and necessity

In the case where $\mathfrak{F}$ is a boolean sub-algebra of $\mathcal{P}(X)$, also probabilities $p: \mathfrak{F} \to [0,1]$ can be taken into account. Once a dual-pair $(N, \pi)$ is given, it appear the problem of which probabilities are consistent with $(N, \pi)$.

The property $N(A) + N(A^C) \leq 1$, equivalent to $N(A) \leq 1 - N(A^C) = \pi(A)$, shows that for all $A \in \mathfrak{F}$ it is

$$N(A) \leq \pi(A),$$
provided the pair \((N, \pi)\) is dual. A probability \(p\) is said \textit{consistent} with the dual pair \((N, \pi)\) if

\[ N(A) \leq p(A) \leq \pi(A), \text{ for all } A \in \mathcal{F}. \]

In this hypothesis, if \(N(A) > 0\), it is also \(p(A) > 0\), and \(\pi(A) > 0\), and if \(\pi(A) = 0\) it is \(N(A) = p(A) = 0\). That is,

- If something is just a little bit necessary, it is probable and possible.
- If something is not possible at all, it is neither necessary nor probable.

Analogously, if \(p(A) > 0\) it is \(\pi(A) > 0\) although it could be \(N(A) = 0\). That is,

- If something is just a little bit probable, it is possible, but not necessarily necessary.

Notice that form \(N(A) \leq p(A) \leq \pi(A)\), or equivalently from \(1 - \pi(A^C) \leq p(A) \leq \pi(A)\), follows \(1 - \pi(A) \leq p(A^C) \leq \pi(A^C)\).

**Example 7.7.1.** Let is \(X = \{1, 2, 3\}\), and \(\mu = 0.7[1 + 1|2 + 0.5|3\). Then, with \(\pi_\mu(A) = \sup_{i \in X} \min(\mu(i), \mu_A(i))\), and \(N_\mu^{(A)} = 1 - \pi_\mu(A)\), we get:

1. \(\pi_\mu(1) = \mu(1) = 0.7, \ \pi_\mu(2) = \mu(2) = 1, \ \pi_\mu(3) = \mu(3) = 0.5.\)
   \(\pi_\mu(\{1, 2\}) = \max(\mu(1), \mu(2)) = 1, \ \pi_\mu(\{1, 3\}) = \max(0.7, 0.5) = 0.7, \)
   \(\pi_\mu(\{2, 3\}) = 1, \ \pi_\mu(X) = 1.\)

2. \(N_\mu(1) = 1 - \pi_\mu(\{2, 3\}) = 1 - 1 = 0, \ N_\mu(2) = 1 - \pi_\mu(\{1, 3\}) = 1 - 0.7 = 0.3, \ N_\mu(3) = 1 - \pi_\mu(\{1, 2\}) = 0\)
   \(N_\mu(\{1, 2\}) = 1 - \pi_\mu(3) = 0.5, \ N_\mu(\{1, 3\}) = 1 - \pi_\mu(2) = 0, \ N_\mu(\{2, 3\}) = 1 - \pi_\mu(1) = 0.3, \ N_\mu(X) = 1.\)

Hence, the consistent probabilities are given by triplets \(p(1), p(2), p(3)\) in \([0, 1]\) such that \(p(1) + p(2) + p(3) = 1\), and verifying:
7.8. PROBABILITY OF FUZZY SETS

\[ 0 \leq p(1) \leq 0.7, \ 0.3 \leq p(2) \leq 1, \ 0 \leq p(3) \leq 0.5. \]

For instance, with \( p(1) = 0.4, \ p(2) = 0.5, \ p(3) = 0.1, \) we have a consistent probability, as well as with \( p(1) = 0.5, \ p(2) = 0.3, \ p(3) = 0.2. \)

But the probability given by \( p(1) = 0.6, \ p(2) = 0.2, \ p(3) = 0.2 \) is not consistent, because of \( p(2) < 0.3. \) With it there is an element whose probability is smaller than its necessity. In the same vein, the probability given by the triplet \( p(1) = 0.1, \ p(2) = 0.3, \ p(3) = 0.6, \) is also non-consistent because one of the probabilities is greater than the corresponding possibility.

7.8 Probability of fuzzy sets

Let’s shortly formalize the classical concept of probability. In a universe \( X, \) let \( \mathcal{F} \subset \mathcal{P}(X) \) be a boolean algebra of parts of \( X, \) that is,

- If \( A, B \in \mathcal{F} \Rightarrow A \cap B, \ A \cup B, \ A^C, \ B^C \in \mathcal{F}. \)
- \( \emptyset \in \mathcal{F}, \ X \in \mathcal{F}. \)

It is said that \( p : \mathcal{F} \rightarrow [0, 1] \) is a probability in \( (X, \mathcal{F}), \) provided

- \( p(\emptyset) = 0 \)
- If \( A \cap B = 0, \) then \( p(A \cup B) = p(A) + p(B). \)

**Theorem 7.8.1.**
1. \( p(A^C) = 1 - p(A), \) for all \( A \in \mathcal{F}. \)
2. If \( A \subset B, \) then \( p(A) \leq p(B) \) (that is \( p \) is a measure)
3. \( p(X) = 1. \)
4. \( p(A \cup B) + p(A \cap B) = p(A) + p(B), \) for all \( A, B \in \mathcal{F}. \)
Proof. Items (1) and (2) just follow from the fact that $p$ is a 0-measure. $p(X) = p(\varnothing^c) = 1 - p(\varnothing) = 1$. Finally, since

$$A \cup B = (A \cap B) \cup (B - A) \cup (A - B),$$

with $(A \cap B) \cap (B - A) = \emptyset$, $(A \cap B) \cap (A - B) = \emptyset$, and $(B - A) \cap (A - B) = \emptyset$, follows

$$p(A \cup B) = p(A \cap B) + p(B - A) + p(A - B).$$

But, from $A = (A - B) \cup (A \cap B)$, and $B = (B - A) \cup (A \cap B)$, it also follows (since the unions are disjunct):

- $p(A) = p(A - B) + p(A \cap B) \Rightarrow p(A - B) = p(A) - p(A \cap B)$
- $p(B) = p(B - A) + p(A \cap B) \Rightarrow p(B - A) = p(B) - p(A \cap B).$

Hence, $p(A \cup B) = p(A \cap B) + p(A) - p(A \cap B) + p(B) - p(A \cap B) = p(A) + p(B) - p(A \cap B).$ \qed

Of special importance for the application are the probabilities defined in the real line $\mathbb{R}$, with $\mathcal{F} = \mathcal{B}$ the so-called Borel’s algebra, given by all the unions, complements and intersections of the open, closed, semi-open, and semi-closed intervals of $\mathbb{R}$. Then, if $A \in \mathcal{B}$, the probability of $A$ is defined by the Lebesgue-Stieltjes integral

$$p(A) = \int_A dP = \int_{\mathbb{R}} \mu_A(x) dx = E(\mu_A),$$

that is, as the mathematical expectation of $\mu_A$. Then, if $\mu \in [0, 1]^\mathbb{R}$ is Borel-measurable, it can be analogously defined

$$p(\mu) = E(\mu) = \int_{\mathbb{R}} \mu(d) dx.$$

Obviously, $p(\mu_0) = E(\mu_0) = 0$, $p(\mu_1) = E(\mu_1) = 1$, and if $\mu \leq \sigma$ follows $p(\mu) \leq p(\sigma)$. In addition, with $\mu \cdot \sigma = \min \circ (\mu \times \sigma)$, $\mu + \sigma = \max \circ (\mu \times \sigma)$, it is
7.8. PROBABILITY OF FUZZY SETS

\[ p(\mu + \sigma) + p(\mu \cdot \sigma) = p(\mu) + p(\sigma), \]

that implies: If \( \mu \cdot \sigma = \mu_0 \), then \( p(\mu + \sigma) = p(\mu) + p(\sigma) \).

Example 7.8.2. Which is the probability of the fuzzy set (fuzzy event) given by

\begin{align*}
10 \cdot p(\mu) &= \int_{[0,10]} \mu dx = \int_{[3,7]} \mu dx = \int_{[3,4]} \mu_1 dx + \int_{[4,6]} 1 dx + \int_{[6,7]} \mu_2 dx = \\
&= 2 + \int_{[3,4]} (x-3) dx + \int_{[6,7]} (7-x) dx = 2 + \frac{1}{2} + \frac{1}{2} = 3. \text{ Then } p(\mu) = \frac{3}{10} = 0.3.
\end{align*}

Example 7.8.3. Which is the probability of the fuzzy event \( \mu \)
Solution

\[ 10 \cdot p(\mu) = \int_{[0,10]} \mu dx = \int_{[4,5]} \mu_1 dx + \int_{[5,6]} \mu_2 dx = \int_{4}^{5} (4 - x) dx + \int_{5}^{6} (6 - x) dx = -\frac{1}{2} + \frac{3}{2} = 1. \] Then \( p(\mu) = 0.1. \)

Example 7.8.4. Which is the probability of the fuzzy event \( \mu : \)

Which is the conditional probability \( p(\sigma/\mu)? \)

Solution

\[ p(\mu) = \int_{[0,10]} \mu dx = \int_{1/4}^{3/4} \mu_1 dx + \int_{3/4}^{1} \mu_2 dx = \frac{1}{4} + \int_{1/4}^{3/4} (2x - \frac{1}{2}) dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \]

\[ p(\sigma|\mu) = \frac{p(\sigma \cdot \mu)}{p(\mu)} = 2p(\sigma \cdot \mu), \] with \( \sigma \cdot \mu = \min \circ (\sigma \times \mu), \) given by

Then \( p(\sigma \cdot \mu) = \int_{1/4}^{3/4} (4 - x) dx + \int_{1/2}^{3/4} x dx = \frac{7}{16}, \) hence \( p(\sigma|\mu) = \frac{7/16}{1/2} = \frac{7}{8}. \)