Asymptotic analysis of the electrostatic wave equation in a toroidal plasma

A. Cardinali and F. Zonca
Associazione EURATOM-ENEA sulla Fusione, Centro Ricerche Frascati, c.p. 65, 00044 Frascati, Italy

(Received 27 May 2003; accepted 7 August 2003)

An analytical study is presented of the wave equation that describes the propagation of an electrostatic pulse in a cold plasma in a general magnetic equilibrium by means of a multiple spatial scale approach. This technique is strictly related with that discussed earlier by Zonca and Chen [Phys. Fluids B 5, 3668 (1993)], and, when applied to plasma instabilities, reduces to the well-known “ballooning formalism” [J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. 40, 396 (1978)]. A simplified equation for the scalar potential in the cold plasma limit will be derived and studied by applying the WKB asymptotic technique to describe the slow radial dependencies of the wave envelope, while the full-wave equation will be considered along the magnetic field lines. This ansatz can be entirely justified on the basis of spatial scale separation in the radial direction and for waves that have parallel group velocity faster than in the perpendicular direction. Thus, this approach could be viewed as a mixed WKB-full-wave technique. © 2003 American Institute of Physics. [DOI: 10.1063/1.1615240]

In the past, full WKB techniques have been applied to solve the equation describing the propagation of a high-frequency wave in a magnetized plasma owing to the intrinsic difficulty to implement a full wave algorithm. The WKB approximation simplifies considerably the problem, but at the same time, in some situations, can lead to an erroneous evaluation of the electric field. Moreover, WKB is unable to study correctly the mode conversion layers and to predict the coupling to other branches of propagating waves. In this Letter, an analytical study of the wave equation which describes the propagation of an electrostatic pulse in a cold plasma and in a general magnetic equilibrium, has been performed by means of a multiple spatial scale approach. Our technique is strictly related with that discussed earlier, and, in special cases discussed below, it reduces to the well-known ballooning formalism. Here, we take this approach further and show how it can actually yield deeper insights into the wave dynamics when viewed within the realm of wave packet propagation. Choosing \( \eta \), the dual variable to \( qR_0k_i = m + nq \) with respect to the Fourier transform (where \( q \) is the safety factor, \( R_0 \) is the major radius of the torus, \( k_i \) is the parallel wave vector, and \( m \) and \( n \) are, respectively, the poloidal and toroidal mode number) as a dimensionless coordinate along the magnetic line of force, we derive an equation for the scalar potential in the cold plasma limit.

As an example of the application of this formalism, we can consider the propagation of a cold electrostatic pulse whose wave equation, as results from the Vlasov–Poisson system, can be written as

\[
\nabla \left[ \left( \frac{e^2}{\epsilon_0} \right) \Phi(\hat{r}) \right] = \nabla \left[ \left( \frac{e^2}{\epsilon_0} \right) \Phi(\hat{r}) \right] = 0. \tag{1}
\]

where \( \epsilon \) is the electric tensor, \( \epsilon_0 \) is the vacuum permittivity, \( \Phi(\hat{r}) \) is the scalar potential, \( \hat{r} \) is a unit vector along the magnetic field line, and \( B \) is the magnetic field. The equation (1) is the relevant equation for the lower hybrid (LH) propagation in tokamak plasmas, and its solution is of fundamental relevance in studying the physics of lower hybrid current drive (LHCD).

To have a suitable expression for the Laplacian in Eq. (1) we choose, as usual, to use a “straight field line” coordinate system \( (r, \phi, \theta) \). The Jacobian is \( J^{-1} = \frac{\psi'(r)}{\nabla r \times \nabla \phi} \), and \( \psi(r) \) is the poloidal magnetic flux, \( r \) is a radial-like flux coordinate, and if we choose the safety factor \( q = (B \cdot \nabla \phi)/(B \cdot \nabla \theta) \) to be a flux surface function, it is possible to write the magnetic field in a compact form as \( B = \psi'(r)(\nabla r \times \nabla \phi + q(r) \nabla \phi \times \nabla r) \). The generalized toroidal angle is \( \phi = \phi - \nu(\theta, r) \), with \( \phi \) the usual toroidal angle, and the function \( \nu(\theta, r) \) must satisfy the condition \( \partial \nu(\theta, r) / \partial \theta = J T(r) |\nabla \phi|^2 - \psi(r) \), where \( T(r) \) is a function related to the plasma current. Note that the choice of the “straight field line” coordinate system enables the quantity \( J B^2 \) to be a flux surface function.

To further simplify notations, from \( (r, \phi, \theta) \) we move to Clebsch coordinates \( (r, \zeta, \theta) \), with \( \zeta = \phi - q \theta \), \( J^{-1} = \frac{\epsilon}{\epsilon_0} \Phi(\hat{r}) \), and the wave equation in the Clebsch coordinate system is given by

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]
In orthogonal coordinate system, the wave equation \( u'' \) becomes

\[
e_{xx} \frac{\partial^2 \Phi(\vec{r})}{\partial \xi^2} + e_{xx} \left[ \nabla^2 \Phi(\vec{r}) \right] = 0.
\]

where \( e_{xx} = 1 \) is a flux function. Moreover, we have considered the ordering \( \nabla \cdot e_{xx} \nabla \Phi = O(n^{-1}) \), with \( n \) the toroidal mode number, much greater than 1. This means that the spatial derivatives of the equilibrium quantities are \( O(n^{-1}) \) smaller than the derivative of the scalar potential, and for this reason negligible. It is worth noting “\( \hat{\theta} \)” in the Clebsh coordinates system, is the magnetic field line coordinate.

If we choose an orthogonal pseudotoroidal coordinate system \((r, \phi, \hat{\theta})\) and assume circular and concentric magnetic flux surfaces, we may explicitly write expressions for \( \hat{\phi} \) and \( \hat{\theta} \). In fact, the generalized poloidal angle \( \hat{\theta} \) is related to the poloidal angle \( \theta \) by the following expression:

\[
\hat{\theta} = 2 \arctan[(\sqrt{1-(r/R_0)^2})(1+r/R_0) \tan(\theta/2)],
\]

while \( v(\hat{\theta}, \theta) = 0 \), and the generalized toroidal coordinate \( \hat{\phi} = \phi \).

\[ e_{xx} \nabla \Phi(\vec{r}) = 0, \]

Equations (4)–(5) preserve the two-dimensional (2D) structure of the original equation, Eq. (1), and they are exact up to \( O(n^{-1}) \).

Formal progress can be made, without loss of generality, assuming the Poisson summation formula (PSF) that allows us to write:

\[
\Phi(\hat{\theta}, \theta) = \sum_m e^{im\hat{\phi}} \Phi_m(\hat{\theta}) = \sum_m \int_{-\infty}^{+\infty} e^{im\hat{\phi}} (\mu - m) \hat{\Psi}(r, \mu) d\mu
\]

\[
= \sum_m e^{im\hat{\phi}} \int_{-\infty}^{+\infty} e^{-im\eta} \hat{\Psi}(r, \eta) d\eta,
\]

where \( \hat{\Psi}(r, \mu) = \Phi_m(\hat{\theta}) \) and

\[
\hat{\Psi}(r, \eta) = \sum_{m} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{im\phi} \hat{\Psi}(r, \eta) d\mu.
\]

Constructing \( \hat{\Psi}(r, \mu) \) such that it is in \( L^2(\mathbb{R}) \) can be done, e.g., either taking

\[
\hat{\Psi}(r, \mu) = \sum_m \Phi_m(\hat{\theta}) T_m(\mu),
\]

or

\[
\hat{\Psi}(r, \mu) = \sum_m \Phi_m(\hat{\theta})(H(\mu - m + \lambda) - H(\mu - m - \lambda)),
\]

with \( 0 < \lambda < 1/2 \), and \( T_m(\mu) = \mu - m + 1, \forall m - 1 \leq m \leq m; T_m(\mu) = m + 1 - m, \forall m \leq m \leq m + 1 \); and \( T_m(\mu) = 0 \) elsewhere, \( H(\mu) \) indicating the Heaviside function.

With \( \hat{\Psi}(r, \mu) \in L^2(\mathbb{R}) \), the following relation holds:

\[
\Phi(r, \hat{\theta}) = 2\pi \sum_m \hat{\Psi}(r, \hat{\theta} + 2\pi m);
\]

thus, the knowledge of \( \hat{\Psi}(r, \eta) \in L^2(\mathbb{R}) \) is completely equivalent to that of \( \hat{\Phi}(r, \hat{\theta}) \) for \( 0 \leq \hat{\theta} < 2\pi \). The equation for \( \hat{\Psi}(r, \eta) \) is easily obtained from Eq. (4), recognizing that \( \partial \hat{\phi} \partial \hat{\theta} = \partial \phi \partial \eta \) and that any periodic function of \( \hat{\phi} \) maps into the same periodic function of \( \eta \) in \( (r, \eta) \) space. Since \( \partial \hat{\phi} \partial \hat{\theta} = \partial \phi \partial \eta \), \( \eta \) is essentially the normalized distance along a magnetic field line. Thus, the 2-D wave equation, Eq. (4), can be reformulated as

\[
\hat{\Psi}(r, \eta) = \frac{e_{xx}}{r^2} \left[ \frac{\psi'}{R_0} \right] \left[ \frac{\nabla \psi^2}{R_0} \right] + \frac{e_{xx}}{r^2} \left[ \frac{\psi'}{R_0} \right] \left[ \frac{\psi}{R_0} \right] + \frac{e_{xx}}{r^2} \left[ \frac{\psi'}{R_0} \right] \left[ \frac{\psi}{R_0} \right] \frac{\psi'}{R_0} - \frac{n^2}{r^2}
\]

\[
\times \Phi(\vec{r}, \hat{\theta}) = 0,
\]

where we have assumed that \( \Phi(\vec{r}) = e^{i\xi + i\phi} \hat{\Phi}(\vec{r}, \hat{\theta}) \). Here \( \hat{\Phi}(\vec{r}, \hat{\theta}) \) being a periodic function of \( \hat{\theta} \), it is possible to introduce the representation for the scalar potential:

\[
\Phi(\vec{r}) = e^{i\xi + i\phi} \hat{\Phi}(\vec{r}, \hat{\theta}) = e^{i\xi + i\phi} \sum_m e^{im\hat{\phi}} \Phi_m(\vec{r}),
\]

for the choice of Eq. (8), while for Eq. (9),

\[
\hat{\Psi}(r, \eta) = \frac{1}{2\pi} \sum_m \frac{e^{im\phi} \Phi_m(\vec{r})}{n^2}.
\]
as it may be easily verified.

The advantage of the formulation of Eq. (11), with b.c.’s set, e.g., by Eq. (12) or (13), become clear only when we assume waves that are characterized by parallel group velocities that are larger than in the perpendicular direction. This is the usual case for \[ |k| \ll |\bar{k}|. \] Then, it is reasonable to assume that parallel wave structures are linear superposition of Eq. (11) solutions, whereas the eikonal ansatz, \( \Psi (r, \eta) \sim \exp(i k_d r) \), is consistent for short-wavelength modes as long as \[ |\partial_r k_d|/k_1^2 \ll 1. \] In other words, with the assumptions \( |k| \gg 1 \) and \( |k| \ll |\bar{k}| \), Eq. (1) can be cast into the form of Eq. (11) and solved as a wave equation along magnetic field lines and as a WKB propagation problem in the radial direction. Introducing the notation \( \Psi_k (r, \eta) \) for the basis function spectrum of \( L_k (\tilde{\eta}, \eta; r, k_{r,k}) \), with

\[
L_k (\tilde{\eta}, \eta; r, k_{r,k}) = \frac{e_{zz}}{\tilde{q}^2 R_0^2} \left( \frac{\partial}{\partial \tilde{q}} \right)^2 + \frac{R_0^2}{\tilde{q}^2} \left( 1 + \frac{\psi^2}{2 T^2} \right) \cdot e_{xx} \left( -k_r^2 \nabla |r|^2 + |\nabla |^2 \eta^2 \right) + 2 k_{r,k} \nabla r \cdot \nabla \nabla \eta \left( \frac{n^2}{R_0^2} \right),
\]

Eq. (11) solutions may be written as

\[
\Psi (r, \eta) = \sum_k A_k (r) \Psi_k (r, \eta),
\]

where \( A_k (r) \sim \tilde{A}_k \exp(i k_d r) \) and the values of \( \tilde{A}_k \) are chosen to match b.c.’s at \( r = a \), e.g., in the form of Eq. (12) or (13). Note that, in general, the basis function spectrum is continuous and Eq. (15) implicitly indicates integration on \( \kappa \).

For each of the \( \Psi_k (r, \eta) \)'s, \( k_{r,k} \) is a solution of the local dispersion function \( D_{\kappa} (\omega, k_{r,k}, r) = 0 \), where

\[
D_{\kappa} (\omega, k_{r,k}, r) = \int_{-\infty}^{+\infty} \Psi_k^* (r, \eta) L_k (\tilde{\eta}, \eta; r, k_{r,k}) \Psi_k \times (r, \eta) \right) \left( \int_{-\infty}^{+\infty} |\Psi_k (r, \eta)|^2 d \eta \right)^{-1}.
\]

As usual, for basis functions in the continuous spectrum, Eq. (16) is defined by taking the expression as a limit of integration on a finite interval.

The *envelope tracing equations* for the radial propagation of the partial amplitudes \( A_k (r) \) are then defined as

\[
\frac{dr}{d\tau} = -\frac{\partial D_k}{\partial k_{r,k}} / \frac{\partial D_k}{\partial \omega}, \quad \frac{dk_{r,k}}{d\tau} = \frac{\partial D_k}{\partial r} / \frac{\partial D_k}{\partial \omega}.
\]

Note that the spectrum of \( \Psi_k (r, \eta) \) depends on the radial position. Thus, Eqs. (15)–(17) with given \( \Psi_k (r, \eta) \) may be used to propagate the \( A_k (r) \)'s over a radial annulus of sufficiently small width. Then the procedure must be iterated using new \( \Psi_k (r, \eta) \)'s and \( A_k (r) \)'s, appropriate for the adjacent annulus. The numerical implementation of this scheme will be discussed in a future paper. Here, we note that the convergence of this procedure is guaranteed by the condition \( |k| \gg 1 \).

Now, we take a step backward and consider Eq. (6) once more, comparing it with the usual “ballooning” representation for low-frequency plasma waves, i.e.,

\[
\dot{\Phi} (r, \theta) = \sum_m e^{im\theta} \Phi_m (r) = A(r) \sum_m e^{im\theta} \int_{-\infty}^{+\infty} e^{-i(nq + m) \eta} \times \Psi (r, \eta) d \eta.
\]

which also implies the analog of Eq. (10), i.e.,

\[
\dot{\Phi} (r, \theta) = 2 \pi A(r) e^{-i(nq) \eta} \sum_m \Psi (r, \theta + 2 \pi m).
\]

The evident analogy is based on fundamental properties of plasma waves when \( |k| \gg 1 \) and \( |k| \ll |\bar{k}| \), where the second condition should be more generally formulated as the requirement of group velocity faster in the parallel than in the perpendicular direction. This fact can be clearly seen from Eq. (15) and the following discussion. In the general problem of radio frequency wave propagation, one deals with the continuous spectrum of the one-dimensional wave operator defined in Eq. (14). Imposing b.c.’s at \( r = a \), then, defines the initial linear superposition of basis functions, whose partial amplitudes obey the *envelope tracing equations*, Eq. (17). For linear stability analyses of short-wavelength modes, on the contrary, one deals with the discrete spectrum of the one-dimensional wave operator. Furthermore, only one discrete eigenmode is considered at a time, and, thus, we may write

\[
\Psi (r, \eta) = A(r) e^{-i(nq) \eta} \Psi (r, \eta),
\]

as it clearly emerges from comparisons of Eqs. (6) and (10) with Eqs. (18) and (19). A byproduct of Eq. (20) is the translational invariance of the poloidal harmonics \( \Phi_m (r) \). Note that, while translational invariance is the physical concept that, in the usual approach, justifies, postulating the “ballooning representation,” Eq. (18), here we have shown that Eq. (18) is a special case of a more general approach based on PSF only. Detailed discussions of these aspects will be presented in a separate work.

Equation (20) has the advantage to explicitly treat the fast radial variation of the mode structure on the \((nq)^{-1}\) scale, typical of short-wavelength plasma eigenmodes but generally not of RF pulses propagating in nonuniform plasmas, due to the *forced* nature of the latter problem. By using Eqs. (18)–(20) in Eq. (11), we obtain the following representation of the wave equation:

\[
\varepsilon_{zz} (r, \omega) \cdot \nabla = \frac{q^2 R_0^2}{r^2} \varepsilon_{ss} (r, \omega) \cdot \nabla \dot{\theta}^2 \times \left( 1 + \frac{r^2 / R_0^2}{q^2 (1 - r^2 / R_0^2)} \right) A(r) \frac{\partial^2 \Psi (r, \eta, \theta)}{\partial \eta^2} - \frac{R^2}{R_0^2} \varepsilon_{zz} (r, \omega) \cdot \nabla \nabla \theta^2 \times \left( 1 + \frac{r^2 / R_0^2}{q^2 (1 - r^2 / R_0^2)} \right) \frac{n^2 q \varepsilon_{ss} (r, \omega)}{r^2} + s^2 (\eta - \theta_k)^2
\]

In Eq. (21), \( \theta_k = k_{1, n q} = -i(nq) / \partial (\partial \partial \partial) \) is an operator that is intended to act on \( A(r) \) only, and \( s = r q_k / q(r) \) is the magnetic shear.
For short-wavelength plasma eigenmodes, Eq. (21) can be solved in ϑ space, assuming, in general, outgoing wave boundary conditions for Ψ(ϑ, r) as ϑ → ∞. For such solution, θk can be effectively treated as a parameter depending on the slow radial variable r only, which corresponds to an eikonal assumption for A(r) in the form A(r) ∼ exp(∫f(q) dϑ). Treating Eq. (21) in ϑ space within such an approach is equivalent to a local full wave solution along the magnetic field lines of the original equation. From a linear superposition of these local waves, it is possible to reconstruct the global solution. The envelope A(r) prescribes amplitude and phases for the appropriate linear superposition of local waves. The governing equation for the radial envelope is obtained multiplying Eq. (21) by Ψ*(ϑ, r) and integrating over the entire ϑ-space. The final equation can be written as

\[ \int_{-∞}^{+∞} d ϑ \Psi^*(ϑ, r) \hat{L}(ω, η; ϑ, θ_k) Ψ(ϑ, r) A(r) = D(ω, θ_k, r) A(r) = 0, \]  

where \( \hat{L}(ω, η; ϑ, θ_k) \) is the linear operator on the left-hand side of Eq. (22). Note that, in Eq. (22), \( θ_k = [-i n q \prime( ϑ ) \times (d / d r)] \) can be still considered as a differential operator due to the spatial scale separation between equilibrium and \( θ_k(0) \) variation, and \( D(ω, θ_k, r) = 0. \) In this way, Eq. (22) is effectively a pseudo-differential equation that can be locally solved within the eikonal approximation. A(r) ∼ exp(∫f(q) dϑ), yielding the ray equations for the envelope phase: Eqs. (17), derived above, with \( θ_k = k_{k, r, q} / n q \prime \) and \( D(ω, k_{k, r, q}) = 0. \) This remark completes our discussion of the analogy of the present approach, based on the PSF, with the BF, showing that our analysis is based only on \( |k_1| \geq 1 \) and \( |k_2| \leq |k_1| \), and includes BF as a particular case.

The advantage of Eqs. (17) on the usual ray tracing formalism is that they are defined in a 2-D phase space \((k_{k, r, q})\) and that the eikonal ansatz has been assumed for the radial dependencies only. Analogously, the equation for short-wavelength eigenmodes can be viewed as an initial value problem for the radial mode structure treated with an eikonal approach, and offer the same advantage of being defined in a 2-D phase space \((k_{k, r, q})\) with respect to similar approaches based on the paraxial WKB description of short-wavelength eigenmodes in a tokamak.

The turning points are identified by the positions of vanishing group velocity, \( ∂D / ∂θ_k = 0 \), and by Eq. (22). The connection expression at the turning point can be obtained by solving

\[ \frac{i}{∂ω} \frac{∂A(r, t)}{∂t} + \frac{∂D}{∂r} (r - r_p) A(r, t) - \frac{1}{2} \frac{∂^2 D}{∂k^2} \left( -\frac{ω n q \prime}{n q} \right)^2 \frac{∂^2 A(r, t)}{∂r^2} = 0, \]  

which is obtained from Eq. (22) when considering the operator \( θ_k \) acting on the function \( A(r, t) \), and where \( r_p \) is the radial position of the turning point.

Considering now that the Laplace transform in time of the function \( A(r, t) \) and introducing the normalized quantity:

\[ a = \left( \frac{r^2 D}{2} \right)^{1/2}, \]  

the variable \( y_ρ = (r - r_p)/a, \) we finally obtain the following equation:

\[ \left[ a^2 A_{w}(y) / (y + Λ) \right] A_{w}(y) + I(y) = 0, \]  

where \( Λ = A/ω d A / d r \), and \( I(y) = I[A(y, t) = 0] / (2πω). \) This equation can be solved in terms of the Airy functions, and it solution gives the behavior of the field near the caustic.

A new approach to study the propagation of a cold electrostatic wave (e.g., lower hybrid wave) with a high toroidal mode number in a simple magnetic field configuration (circular and concentric magnetic surfaces) has been proposed that relies on a multiple spatial scale analysis and on the Poisson summation formula. The present approach actually applies in far more general cases than that discussed here, and it is not limited to either electrostatic waves (cf., e.g., Ref. 2) or to simplified geometry (cf., e.g., Ref. 4). In fact, the validity of the present approach relies simply on the possibility of a systematic asymptotic expansion based on the existence of separation scales, i.e., on the high mode number of the considered wave. More precisely, the validity limits are \( |k_1| \geq 1 \) and \( |k_2| \leq |k_1| \), where the second condition should be more generally formulated as the requirement of group velocity faster in the parallel than in the perpendicular direction. Here we have chosen to discuss a simple test but still relevant application for the sake of clarity.

A wave equation has been derived for the scalar potential by introducing the normalized length along the magnetic line of force η as a new variable coordinate, while the operator \( θ_k = k_{k, q} / n q \prime \) acting on the amplitude \( A(r) \) can be considered as a parameter (the normalized radial wave vector) when the full 2-D PDE is reduced to two “nested” 1-D problems. Once solved, the lowest level 1-D ODE along the field line coordinate yields to a local dispersion relation connecting \( r \) to the normalized radial wave vector \( θ_k \), which can be obtained and numerically studied. To determine the radial dynamics of the wave, a set of two ray tracing equations along the slow radial variable “r” are derived and discussed. In particular, the turning point position can be determined, as well as the reflection and transmission conditions of rays at these points.