The fishbone burst cycle*

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Observation of fishbone oscillations

- Experimental observation of fishbones in PDX [McGuire et al. 83] with macroscopic losses of injected fast ions ...
Followed by numerical simulation of the **mode-particle pumping** (secular) loss mechanism [White et al 83] ...

\[ \dot{\omega} \approx \omega_B^2 \]

... and the theoretical explanation of the **resonant internal kink excitation by energetic particles** and the (model) dynamic description of the fishbone cycle [Chen, White, Rosenbluth 84]
The fishbone dispersion relation

The problem was solved by [Chen, White, Rosenbluth 84]: first example of the General Fishbone Like Dispersion Relation (GFLDR; \( n = 1 \))

\[ i|s|\Lambda_n = \delta \hat{W}_{nf} + \delta \hat{W}_{nk} . \]

Definitions: drop subscript \( n \) for simplicity, \( \Delta q_0 = 1 - q_0 \),

\[ \Lambda^2 = (\omega/\omega_A^2)(\omega - \omega_{*pi}) (1 + 0.5q^2 + 1.6q^2(R_0/r)^{1/2}) \]

[Graves et al. 2000]

\[ \delta \hat{W}_f = 3\pi \Delta q_0 \left( 13/144 - \beta_{ps}^2 \right) \left( r_s^2/R_0^2 \right) \; ; \; \beta_{ps} = -(R_0/r_s)^2 \int_0^{r_s} r^2(d\beta/dr)dr \]

[Bussac et al. 1975]

\[ \delta \hat{W}_k = \frac{2}{r_s^2} \int_0^{r_s} r^3 dr \int \xi d\xi d\lambda \sum_{v_\parallel/|v_\parallel|=\pm} \frac{\pi^2 R_0 e^2}{c^2 q \tau_b \omega_d^2 m} \left( \frac{\tau_b \omega_d^2}{\omega_0(\tau)} \right) \int_+^{+\infty} \frac{\omega + \omega_0(\tau)}{\omega_d - \omega_0(\tau) - \omega} e^{-i\omega t} Q_{k,\omega_0(\tau)} \hat{F}_0(\omega) d\omega \]

\[ \xi = v^2/2 \; , \; \lambda \equiv \mu B_0/\xi \; \; \; Q_{k,\omega_0(\tau)} \hat{F}_0 \equiv (\omega_0(\tau)\partial_\xi + \Omega^{-1}k \times b \cdot \nabla) \hat{F}_0 \]
General Fishbone Like Dispersion Relation (GFLDR)

\[ i|s|Λ(ω) = \delta \hat{W}_f + \delta \hat{W}_k(ω) \]

- \( s = \text{magnetic shear} \)

- \( Λ(ω) \): “inertia” (kinetic energy) due to background plasma
  \( \Rightarrow \) Structures of continuous spectrum, gaps, and resonant absorption

- \( \delta \hat{W}_f \): “\( \delta W \)” (potential energy) due to background plasmas
  \( \Rightarrow \) existence of discrete AEs (different types; depending on equilibrium)

- \( \delta \hat{W}_k \): “\( \delta W \)” (active potential energy) due to EPs
  \( \Rightarrow \) instability mechanisms & new unstable modes:
  Energetic Particle continuum Modes (EPMs)/Fishbones [Chen 1994]

For EPM/fishbone \( \Rightarrow iΛ \) represents continuum damping [Chen et al 84]

\[ \Re \delta \hat{W}_k(ω_r) + \delta \hat{W}_f = 0 \quad \Rightarrow \text{determines} \; ω_r \; , \; \text{(non - perturbative)} \]

\[ \gamma/ω_r = (-ω_r\partial_{ω_r} \Re \delta \hat{W}_k)^{-1}(\Im \delta \hat{W}_k - |s|Λ) \quad \Rightarrow \text{determines} \; γ/ω_r \]
Outstanding questions

- **Why does it chirp?**
  ⇒ connected with nature of wave-EP interaction, and mode dispersion relation (non-perturbative)

- **What is the frequency sweeping rate?**
  ⇒ connected with wave-EP power exchange, and self-consistent (non-perturbative) interplay of mode structure and EP transport

- **What is the mechanism of EP loss?**
  ⇒ connected with non-adiabatic frequency sweeping and phase-locking
Why does it chirp?

The diagram of the process is defined in the top frame, while the solution of the "Dyson" equation corresponds to the summation of all terms in the Dyson series (bottom).

Generation of the distribution $\delta f_k$ due to the interaction of $f_0$ with $\delta \phi_k$, corresponding to the solution of the GFLDR.

Nonlinear distortion of $f_0$ due to emission and absorption of the field $\delta \phi_k$. 

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Nonlinear distortion of $f_0$ is accompanied by **self-consistent nonlinear evolution of mode structure** (frequency), due to the **non-perturbative wave-EP interaction** ⇒ **frequency chirping**.

Not all modes can chirp: fishbone **real frequency** (dispersion relation) is set by the characteristic EP precession rate, which depends on radial position ⇒ **continuum of possible fishbone frequencies** controlled by EP radial profile; dominated by the **strongest growing mode**.

Non-perturbative EP response (dispersion relation), via the selection of the strongest growing mode (maximization of wave-EP power transfer) also dictates the optimal chirping rate for preserving the resonance condition throughout the nonlinear phase ⇒ **phase-locking** [C&Z RMP16]:

$$\omega \simeq \omega_d \quad |\dot{\omega}| \simeq \left| \delta v_r \frac{\partial}{\partial r} \bar{\omega}_d \right| \simeq \left| \frac{nq}{r} \frac{c}{B_0} \delta \phi \frac{\partial}{\partial r} \bar{\omega}_d \right| \propto \omega_B^2$$

**non-adiabatic frequency sweeping**
Nonlinear wave-EP interactions are accounted for via \((n = 1)\) Dyson equation

\[
\hat{F}_0(\omega) = \frac{i}{\omega} S_\tau \hat{F}_0(\omega) + \frac{i}{\omega} \hat{S}(\omega) + \frac{i}{2\pi \omega} \tilde{F}_0(0) + \frac{e}{m \omega} \frac{nc}{\omega} \partial_r \left\{ \frac{Q^*_{k0,\omega_0(\tau)}}{\omega_0^*(\tau)} \right. \\
\times \frac{\hat{F}_0(\omega - 2i\gamma_0(\tau))}{\omega - \omega_0(\tau) + n\bar{\omega}_{dk0}} + \frac{Q_{k0,\omega_0(\tau)}}{\omega_0(\tau)} \frac{\hat{F}_0(\omega - 2i\gamma_0(\tau))}{\omega + \omega_0^*(\tau) - n\bar{\omega}_{dk0}} \hat{\omega}_{dk0} \left| \delta \bar{\phi}_{k0}(r, \tau) \right|^2 \left\} 
\]

Solution strategy of the nonlinear problem (small fluctuation amplitude expansion):

- At the lowest order, the problem is solved satisfying the linear dispersion relation
- At next order the nonlinear dynamics describes the mode amplitude and frequency evolution; and EP transport
Introduce the radial displacement

\[ \delta \xi_{rk0}(r, \tau) = \frac{k_0 c}{\omega_0 B_0} \delta \phi_{k0}(r, \tau) = -\frac{nqc}{\omega_0 r B_0} \delta \phi_{k0}(r, \tau). \]

From the Dyson Equation, assume \(|\omega_{*EP}| \gg |\omega_0| \Rightarrow \omega_0(\tau)^{-1} Q_{k0,\omega_0(\tau)} \hat{F}_0 \simeq (\omega_0 \Omega)^{-1} (-nq/r) \partial_r \hat{F}_0\). Furthermore, \(d\psi/dr = B_0(r/q)\). Then, by direct substitution, at the two lowest orders in \(\gamma_0/\omega_0\)

\[ \hat{F}_0(\omega) = \frac{i}{\omega} S \hat{F}_0(\omega) + \frac{i}{\omega} \hat{S}(\omega) + \frac{i}{2\pi \omega} \hat{F}_0(0) - \frac{2i}{\omega} \left( \frac{n \bar{\omega}_{dk0}}{\omega_0 r} \right) \frac{\partial}{\partial r} \]

\[ \times \left[ \frac{(\gamma_0 + i\omega) + (n \bar{\omega}_{dk0} - \omega_0 r)(\gamma_0/\omega_0 r)}{(n \bar{\omega}_{dk0} - \omega_0 r)^2 + (\gamma_0 + i\omega)^2} |\delta \xi_{rk0}(r, \tau)|^2 \frac{\partial}{\partial r} \hat{F}_0(\omega - 2i\gamma_0(\tau)) \right]. \]

This expression has to be employed within the \(\delta \hat{W}_k\) expression.
Solution of Dyson equation reveals [C&Z NJP15; RMP16]:

- **phase locking**: resonance condition preserved throughout the nonlinear phase
- **phase bunching**: periodic modulation due to EP phase synchronization caused by resonance tuning & detuning (non-adiabatic chirping)

Numerical solution is required in general. Analytical progress is possible assuming:

- considering deeply trapped EP with radially localized interaction with internal kink-like mode structure
- describing the **average evolution of the phase space zonal structure** (PSZS) [C&Z NJP15; RMP16]
Nonlinear fishbone dynamics (I)

Thanks to these assumptions we can define resonant particle $\beta_E$ as

$$\beta_{Er}(r; \omega_0(\tau)) = 2\frac{\pi^2}{B_0^2} m |\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v\parallel/|v\parallel|=\pm1} \tau_b \bar{\omega}_d^2$$

$$\times \int_{-\infty}^{\infty} \frac{(\gamma_0 - i\omega)}{(\bar{\omega}_d - \omega_0r)^2 + (\gamma_0 - i\omega)^2} e^{-i\omega t} \hat{F}_0(\omega) d\omega .$$

With this expression,

$$\text{Im} \delta \hat{W}_k = -\frac{R_0}{r_s} \int_0^{r_s} q^2 r \left( \frac{R_0}{r} \right)^{1/2} \frac{\partial}{\partial r} \left[ \left( \frac{r}{R_0} \right)^{1/2} \beta_{Er}(r; \omega_0(\tau)) \right] dr$$

$$= \frac{R_0}{r_s} \int_0^{r_s} \left[ -r q^2 \frac{\partial \beta_{Er}}{\partial r} - q^2 \beta_{Er}/2 \right] \frac{dr}{r_s} ,$$

Resonant particle $\beta_E$ describes the PSZS $\beta_E$ that controls wave-particle power exchange.
In the same way, we can define non-resonant particle $\beta_E$ as

$$\beta_E(r; \omega_0(\tau)) = 2 \frac{\pi^2}{B_0^2} m |\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1} \tau_b \bar{\omega}_d^2$$

$$\times \int_{-\infty}^{\infty} \frac{(\bar{\omega}_d - \omega_0 r)}{(\bar{\omega}_d - \omega_0 r)^2 + (\gamma_0 - i\omega)^2} e^{-i\omega t} \hat{F}_0(\omega) d\omega .$$

With this expression,

$$\Re \delta \hat{W}_k \simeq \Re \delta \hat{W}_k^L = - \frac{R_0}{r_s} \int_0^{r_s} q^2 \frac{r}{r_s} \left( \frac{R_0}{r} \right)^{1/2} \frac{\partial}{\partial r} \left[ \left( \frac{r}{R_0} \right)^{1/2} \hat{\beta}_E(r; \omega_0(\tau)) \right] dr ,$$

Non-resonant particle $\beta_E$ describes the PSZS $\beta_E$ that controls mode frequency.
Nonlinear fishbone dynamics (II)

- As reference case, consider a simple isotropic slowing down distribution,

\[
\bar{F}_0 = \frac{3P_0E}{4\pi E_F} \frac{H(E_F/m_E - \mathcal{E})}{(2\mathcal{E})^{3/2} + (2E_c/m_E)^{3/2}},
\]

where \( H \) denotes the Heaviside step function and the normalization condition is chosen such that the EP energy density is \((3/2)P_0E\) for \( E_F \gg E_c \), and EP energy is predominantly transferred to thermal electrons by collisional friction as it occurs for \( \alpha \)-particles in fusion plasmas.

- Reconsider the expressions of \( \Re\delta \hat{W}_k \) and \( \hat{\beta}_E(r; \omega_0(\tau)) \); and

\[
\delta \hat{W}_f + \Re\delta \hat{W}_{\gamma k} \simeq 0.
\]

- For \( \Re\Lambda^2 > 0 \), this equation (real part of GFLDR) determines the real frequency of the fishbone mode for \( |\gamma_0/\omega_0| \ll 1 \) [Chen 1984].
The fishbone frequency is set by the condition $\omega_0/\bar{\omega}_d F \simeq \text{const}$, to be computed at the position of the radial shell where the most significant EP contribution is localized.

The GFLDR (imaginary part; for $|\gamma_0/\omega_0| \ll 1$), gives the evolution of the fishbone amplitude

$$\gamma_0 \left( -\partial \text{Re} \delta \hat{W}_k^L / \partial \omega_0 r \right) = \text{Im} \delta \hat{W}_k - |s| \Lambda(\omega_0 r) .$$

This equation expresses the competition between EP drive and continuum damping; and can be cast as

$$\frac{\partial}{\partial t} \ln |\delta \xi_0|^2 = \frac{2(R_0/r_s)}{-\partial \text{Re} \delta \hat{W}_k^L / \partial \omega_0 r} \left\{- \int_0^{r_s} q^2 \frac{r}{r_s} \left( \frac{R_0}{r} \right)^{1/2} \right. \\
\left. \times \frac{\partial}{\partial r} \left[ \left( \frac{r}{R_0} \right)^{1/2} \beta_{Er}(r; \omega_0(\tau)) \right] dr - \left( \frac{r_s}{R_0} |s| \Lambda(\omega_0 r) \right) \right\} .$$
The simplest way to close the (GFLDR) evolution for the fishbone amplitude is to obtain the evolution equation for $\beta_{Er}(r; \omega_0(\tau))$ directly from the Dyson equation for $\hat{F}_0(\omega)$.

Calculation of $\delta \hat{W}_k$ and $\beta_{Er}(r; \omega_0(\tau))$ involves a vel. space and freq. integral

$$\propto \left[ \frac{(\gamma_0 + i\omega) + (n\bar{\omega}_{dk0} - \omega_0r)(\gamma_0/\omega_0r)}{(n\bar{\omega}_{dk0} - \omega_0r)^2 + (\gamma_0 + i\omega)^2} \right] \left| \delta \bar{\xi}_{rk0}(r, \tau) \right|^2 \frac{\partial}{\partial r} \hat{F}_0(\omega - 2i\gamma_0(\tau))$$

Calculation can be done noting, for $|a|, |b| \ll 1$, $\Re a > 0$ and $\Re b > 0$

$$\frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \rightarrow \frac{\pi}{a + b} \delta(x) \quad \frac{1}{(x^2 + a^2)(x^2 + b^2)} \rightarrow \frac{\pi}{ab(a + b)} \delta(x)$$

Recall that $\left| \delta \bar{\xi}_{rk0}(r, t) \right|^2 = \exp(2\gamma_0t) \left| \delta \bar{\xi}_{rk0}(r, \tau) \right|^2$. For strongly growing modes, $\gamma_0 \sim |\omega| \Rightarrow \gamma_0 \sim \tau_{NL}^{-1}$, as for SAW excited by EP in tokamaks.
Nonlinear fishbone equations

With the previous result, the nonlinear equation for $\beta_{Er}(r; \omega_0(\tau))$ is [C&Z RMP16]

$$\partial_t \beta_{Er} = \left( \beta_{ErS} - \nu_{ext} \beta_{Er} \right) + \partial_t^{-1} \left( \frac{R_0}{r} \right)^{1/2} \left\{ \frac{q}{r} \frac{\partial}{\partial r} \left[ \frac{r}{q} |\omega_0|^2 |\delta \xi_{r_0}|^2 \frac{\partial}{\partial r} \left( \frac{r}{R_0} \right)^{1/2} \beta_{Er} \right] \right\}.$$ 

From external source and collision operator, plus the definition of $\beta_{Er}(r; \omega_0(\tau))$, we have

$$\dot{\beta}_{ErS} \equiv \frac{2 \pi^2}{B_0^2} m |\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_{||}/|v||=\pm 1} \tau_b \bar{\omega}_d^2 \frac{\gamma_0}{(\bar{\omega}_d - \omega_{r_0})^2 + \gamma_0^2} S(t),$$

$$\nu_{ext} \beta_{Er} \equiv -2 \frac{\pi^2}{B_0^2} m |\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_{||}/|v||=\pm 1} \tau_b \bar{\omega}_d^2 \frac{\gamma_0}{(\bar{\omega}_d - \omega_{r_0})^2 + \gamma_0^2} St F_0(t),$$
Resonant EPs convect outward with radial speed $|\delta u_n| \Rightarrow$ Nonlinear saturation occurs when $|\delta u_n|/\gamma_L \sim r_s$

$r_s \sim$ mode structure width $\Rightarrow$ Wave-EP interaction domain

Consistent with numerical simulation results by [GY Fu et al POP 2006].

Near marginal stability regime explored by [M. Idouakass et al POP16; 2017 tbs] analytically and numerically

Electron-fishbone simulation results with HMGC code [Vlad et al NF13]; [Vlad et al NJP16].
What is the frequency sweeping rate?

- Connected with wave-EP power exchange, and self-consistent (non-perturbative) interplay of mode structure and EP transport
- Determined by the phase locking condition, $\omega \approx \tilde{\omega}_d$, and by the PSZS evolution equation for $\beta_{Er}$

$$\partial_t \beta_{Er} = \left( \dot{\beta}_{ErS} - \nu_{ext}\beta_{Er} \right) + \partial_t^{-1} \left( \frac{R_0}{r} \right)^{1/2} \left\{ \frac{q}{r} \frac{\partial}{\partial r} \left[ \frac{r}{q} |\omega_0|^2 |\delta \xi_0|^2 \frac{\partial}{\partial r} \left( \left( \frac{r}{R_0} \right)^{1/2} \beta_{Er} \right) \right] \right\} .$$

- By balancing linear and nonlinear terms, the PSZS radial propagation speed is the instantaneous $E \times B$ velocity.
- Corresponding frequency chirping is non-adiabatic: $\dot{\omega}_0 \lesssim \omega_B^2$ 
  $\Rightarrow$ EP transport due to resonance tuning/detuning $\oplus$ radial decoupling.
What is the mechanism of EP loss?

- **Mode particle pumping** [White et al 83, Chen et al 84]
  ⇒ Connected with non-adiabatic frequency sweeping and phase-locking

- **Non-adiabatic autoresonance** [C&Z NJP15; RMP16]
  - EPs amplify fishbone over a finite interaction time, $\tau_I$, as they are convected outward
    \[ \int_0^{\tau_I} (\omega_{0r}(\tau) - \bar{\omega}_d) d\tau \simeq \pi \]
  - PSZS slips over EP population and amplification continues (resonance tuning/detuning) until particles are radially decoupled
  - **Strongly driven fishbone saturation** [Fu et al 06; Vlad et al 13,16] corresponds to radial decoupling within $\tau_I$ [Near marginal stability studied by M. Idouakass et al 2017 tbs]
  - **Similar to amplification of a short FEL pulse** [Bonifacio et al. 90,94] (finite interaction time fishbone $\Leftrightarrow$ finite interaction length FEL pulse)
FEL amplification and fishbones

[Giannessi et al, 2005]

Profile of the radiation pulse vs. the longitudinal electron beam coordinate

- Steady state FEL theory: radiation slips forward over the electron beam as required by resonance condition
- Slippage effect $\Rightarrow$ finite cooperation length over which electrons can emit with longitudinal coherence

Longitudinal phase space
Reduced nonlinear equations: predator-prey model

- Original work on fishbone proposed a qualitative model for fishbone cycle ⇒ secular EP loss mechanism.

- This understanding implies $|\delta \xi_{r0}| \sim r_s |\gamma_L/\omega_0|$

- From [Chen et al. 1984], fishbone cycle can be described as
  \[
  \frac{d\beta}{d\tau} = S - A\beta_c ,
  \]
  \[
  \frac{dA}{d\tau} = \gamma_0 (\beta/\beta_c - 1) A .
  \]

- This system of equations can be obtained from our NL fishbone equations.
  - the first one from $\beta_{Er}(r;\omega_0(\tau)) \rightarrow \beta$ equation
  - the second one from the fishbone amplitude evolution equation

Here, $\tau$ is a normalized time, $A = |\delta \xi_{r0}|/r_s$ is the normalized fishbone amplitude and $\gamma_0$ is a measure of the linear growth rate. Furthermore, $\partial_t^{-1} \sim \tau_{NL} \sim r_s/(|\omega_0||\delta \xi_{r0}|)$ in the $\beta_{Er}(r;\omega_0(\tau))$ evolution equation; and $\partial_r^2 \sim -1/r_s^2$. [C&Z RMP16]
The solution of the predator-prey model is cyclic; i.e., it can be generally written as $F(A, \beta) = \text{const}$, where $F(A, \beta)$ has a maximum at the fixed point position $\beta = \beta_c$, $A = S/\beta_c$.

A crucial feature of the model is the linear dependence on $A$ of the loss term in the $\beta$ evolution equation.

$\Rightarrow$ Manifestation of secular resonant EP losses by mode particle pumping [White et al 1983].

It illuminates the role of sources and collisions as well as their link to the fluctuation strength: $S \leftrightarrow \dot{\beta}_{ErS} - \nu_{ext}\beta_{Er}$

$$\dot{\beta}_{ErS} \equiv 2\frac{\pi^2}{B^2_0} m |\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1} \tau_b \bar{\omega}_d^2 \frac{\gamma_0}{(\bar{\omega}_d - \omega_0)^2 + \gamma_0^2} S(t) ,$$

$$\nu_{ext}\beta_{Er} \equiv -2\frac{\pi^2}{B^2_0} m |\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1} \tau_b \bar{\omega}_d^2 \frac{\gamma_0}{(\bar{\omega}_d - \omega_0)^2 + \gamma_0^2} StF_0(t) .$$
Conclusions

- Nonlinear theory of fishbone excitation by EP is at hand [C&Z RMP16] and supports the qualitative model for fishbone cycle proposed originally [Chen et al 1984].

- Present nonlinear theory can be extended to include generation of zonal structures; i.e., wave-wave and wave-particle interactions on the same footing. Work is in progress [L. Chen, Z. Qiu and F. Zonca].

- Interesting mutual and positive feedbacks of theory development with numerical simulation results.