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Theory on excitations of drift Alfvén waves by energetic particles. II. The general fishbone-like dispersion relation

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The theoretical framework of the general fishbone-like dispersion relation (GFLDR), presented and discussed in the Companion Paper [Phys. Plasmas 21, 072120 (2014)], is applied to cases of practical interest of shear/drift Alfvén waves (SAWs/DAWs) excited by energetic particles (EPs) in toroidal fusion plasmas. These applications demonstrate that the GFLDR provides a unified approach that allows analytical and numerical calculations of stability properties, as well as mode structures and, in general, nonlinear evolutions, based on different models and with different levels of approximation. They also show the crucial importance of kinetic descriptions, accurate geometries and boundary conditions for predicting linear as well as nonlinear SAW/DAW and EP behaviors in burning plasmas. Thus, the GFLDR unified theoretical framework elevates the interpretative capability for both experimental and numerical simulation results. [http://dx.doi.org/10.1063/1.4889077]

I. INTRODUCTION

In this work, we adopt the general theoretical framework introduced in Ref. 1 and present detailed applications of the general fishbone-like dispersion relation (GFLDR)$^{2-9}$ to investigations of shear and drift Alfvén waves (SAWs/DAWs) excited by energetic particles (EPs) in toroidal fusion plasmas. This unified description can be adopted to understand and explain a broad spectrum of SAW/DAW fluctuations of practical interest; i.e., those that are characterized by two distinct radial scales, one of which much shorter than the equilibrium scale length. Therefore, it elevates the interpretative capability for both experiments and numerical simulations.$^{10}$

The GFLDR is global by construction$^1$ and is based on the nonlinear gyrokinetic quasineutrality condition and vorticity equation. The corresponding theoretical framework can thus, be adopted for stability analyses of macroscopic SAW/DAW modes excited by EPs as well as their nonlinear dynamics, saturation, and ensuing transports. In the general case, mode structures are assumed as given by numerical solution. However, analytical expressions of mode structures could be obtained in special cases from the underlying gyrokinetic equations. Furthermore, the variational nature of the integral formulation suggests the possibility of adopting a trial function approach for both stability analyses as well as for derivations of simplified nonlinear descriptions of mode amplitude initial value problems, which could often be cast as nonlinear Schrödinger equations.$^{1,10}$

The GFLDR can be viewed as a kinetic energy principle,$^{1,12}$ which recovers various formulations of the MHD energy principle$^{13-20}$ in the proper limits.$^1$ Its form reflects the two spatial scale nature of SAW, due to the existence of the SAW continuous spectrum in nonuniform toroidal plasmas, which generally consist of singular (inertial) and regular (ideal MHD) structures. In this way, the GFLDR helps identifying spatial scales on which different physics manifest themselves and determine observable properties of the SAW/DAW fluctuation spectrum with the corresponding characteristic temporal scales.$^{10}$ The general processes underlying the excitation of SAW/DAW by EPs in toroidal plasmas are discussed in Ref. 1, with emphasis on linear as well as nonlinear dynamics in inertial and ideal regions. The present work is limited to linear SAW/DAW stability and mode structure analyses; thus, the GFLDR adopted here deals with a single toroidal mode number. However, the GFLDR can be extended to multiple toroidal mode numbers; casting it as system of nonlinear Schrödinger equations involving mode-mode couplings.$^{1,10,21}$

In the present work, we demonstrate the crucially important role of equilibrium geometry, plasma nonuniformities, and kinetic descriptions by direct calculation of the GFLDR for fishbones$^5$ (cf. Sec. III), low frequency SAWs (cf. Sec. IV), toroidal Alfvén eigenmodes$^22$ (TAEs; cf. Sec. V); and energetic particle modes$^4$ (EPMs; cf. Sec. VI). The calculation, where possible, adopts analytical methods and establishes a close link with well-known results obtained previously. The important element of novelty of the present approach stands in the systematic use of the unified GFLDR theoretical framework, which shows, step by step, where calculations can be performed either analytically, with various levels of approximation, or numerically; adopting different physical models and solution methods. As a result, the GFLDR not only provides a framework for the unified description of a broad spectrum of SAW/DAW fluctuations of practical interest but also for numerical code benchmarking and verification and validation of simulation results against experimental observations.

For the readers’ convenience, Sec. II gives a synthetic summary of practically most useful elements of the

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GFLDR theoretical framework described in detail in Ref. 1. Sections III–VI are then devoted to detailed application of the GFLDR to various branches of the SAW/DAW fluctuations spectrum and to the description of their properties in terms of the underlying physics. Section VII, meanwhile, presents examples of experimental observations where the GFLDR theoretical framework could be used to provide more insights to the underlying physics. Concluding remarks and discussions are finally given in Sec. VIII.

II. THE GFLDR THEORETICAL FRAMEWORK

The GFLDR is global by construction and generally reads as

\[ i[s] A_n = \delta \dot{W}_{nf} + \delta \dot{W}_{nk} \]  

(1)

Here, \( s \) denotes magnetic shear at the kinetic/singular layer, the \( \sim \) \( \Lambda_n \) term accounts for the layer response, while the right hand side comes from core plasma and EP contributions in the regular ideal regions (cf. Sec. I). Furthermore, the \( n \) subscript stands for the reference toroidal mode number, magnetic shear is defined as

\[ s = s(r) = nr'(r)/q(r), \]  

(2)

\( q(r) \) is the safety factor describing the pitch of equilibrium magnetic field lines

\[ q(r) = B_0 \cdot V_\zeta / B_0 \cdot V_\theta, \]  

(3)

\( B_0 \) is the equilibrium magnetic field, and we have adopted straight magnetic field line toroidal flux coordinates \( (r, \theta, \zeta) \), with \( r \) the radial “magnetic flux” variable; and \( \theta \) and \( \zeta \) periodic angular coordinates along poloidal and toroidal directions, respectively. In the following, we assume tokamak plasma equilibria for which \( \zeta \) is the ignorable (symmetry) variable. Equation (1) is written for a single toroidal mode number, \( n \), but, in the general case, should be intended as the \( n \)-th row of a matrix equation for the complex amplitudes \( \cdots \text{A}_n, \cdots \) involving nonlinear mode couplings, which can be cast as nonlinear Schrödinger equation.\(^{1,10,21} \) In the linear limit, considered here, each individual \( n \) is independent; and Eq. (1) becomes the dispersion relation of the corresponding fluctuation.

The inertial layer response accounts for plasma inertia enhancement due to both toroidal geometries as well as kinetic effects.\(^{1} \) In general, it can be computed from the asymptotic solution of the gyrokinetic vorticity equation in the extended poloidal angle \( \vartheta \)-space. Introducing the mapping \( \mathcal{P}_0(r, \vartheta) : f(r, 0, \zeta) \mapsto f_n(r, \vartheta) \) of a generic fluctuating field \( f(r, 0, \zeta) \) (time dependence is suppressed here to simplify notation) to \( f_n(r, \vartheta) \),\(^{1,23} \)

\[ f(r, 0, \zeta) = 2\pi \sum_{m,n \in \mathbb{Z}} e^{-i(m-nq)(\theta-2\pi)} \tilde{f}_n(r, \vartheta) \]

\[ = \sum_{m,n \in \mathbb{Z}} e^{i(m-nq)\vartheta} \tilde{f}_n(r, \vartheta) d\vartheta, \]

\[ = \sum_{m,n \in \mathbb{Z}} e^{i(m-nq)\vartheta} \mathcal{P}_0(r, \vartheta) [f] d\vartheta, \]  

(4)

we can represent SAW/DAW fluctuations in the \( \vartheta \)-space by two fluctuating scalar fields, \( \delta \phi_n \) and \( \delta \psi_n \), related by Eq. (4) to scalar \( \delta \phi \) and vector potential fluctuations \( \delta \psi \equiv -c \delta \zeta^{-1} b \cdot V \delta A \), with \( b = B_0/B_0 \). Furthermore, let us consider a tokamak equilibrium with

\[ B_0 = F(\psi) V \phi + V \phi \times V \psi, \]  

(5)

where \( \phi \) is the physical toroidal angle and \( \psi(r) \) is the magnetic flux function. Note also that multiplication by a periodic function \( \vartheta \) corresponds to multiplication by a periodic function \( \vartheta \) in \( \vartheta \)-space; and that \( B \cdot V \mapsto (J B_0)^{-1} \partial \vartheta, \) with \( J = (V_\psi \times V_\theta \cdot V \varphi)^{-1} \) the Jacobian of the considered coordinate system. Then, the large-\( |\vartheta| \) (short radial scale) asymptotic form of the gyrokinetic vorticity equation in \( (r, \vartheta) \) space is\(^{1} \)

\[ \frac{\partial}{\partial \vartheta} \left[ \frac{c^2 k_0^2 k_\perp \partial}{4\pi J^2 B_0^2} \left( \frac{\partial^2}{\partial \vartheta^2} - \frac{\partial^2}{\partial r^2} \right) V_\perp \right] \delta \Psi_n = \frac{J^2 B_0^2}{V_\perp^2} \frac{\partial}{\partial \vartheta} \partial \psi_n + 2 \frac{k_0^2 k_\perp^2}{c^2} \frac{\partial}{\partial \vartheta} \left[ \frac{\partial}{\partial \vartheta} + i \omega_{spi} + i \omega_{sTi} \right] \delta \Phi_n, \]  

(6)

where \( k_0 = -nq/r, \delta \Psi_n \equiv k_\perp \delta \psi_n, \delta \Phi_n \equiv k_\perp \delta \phi_n, \) and \( k_0^2 k_\perp^2 = -\nabla_\perp^2, \) with

\( \nabla_\perp \mapsto V_\perp (k_0 s \partial_r + \partial_\vartheta + i n \partial_\zeta + V_\zeta (\partial_r - i n q) - (J B_0)^{-1} b \partial_\vartheta. \)  

(7)

Here, we have also introduced the magnetic curvature \( k = b \cdot V b, J_0 \) is the Bessel function of argument \( \lambda, \lambda^2 = 2nB_0 k_\perp^2 / \Omega^2, \Omega = c B_0 / mc \) is the cyclotron frequency, \( \mu = v_\perp^2 / (2B_0) + \cdots \) is the magnetic moment, \( \perp \) and \( \| \) subscripts denote perpendicular and parallel components with respect to \( b, (\cdots)_\perp \) denotes velocity space integration; and \( \omega_{spi} = \omega_{spi} + \omega_{sTi} \) is the thermal ion diamagnetic frequency

\[ \omega_{sTi} = \frac{T_0 c}{e n B_0} \left( b \times V_\perp n_0 \right) \cdot k_\perp, \]  

(8)

\[ \omega_{spi} = \frac{T_0 c}{e n B_0} \left( b \times V T_0 \right) \cdot k_\perp, \]  

with \( k_\perp = -\nabla_\perp. \) Meanwhile, the non-adiabatic particle response \( \delta \psi_n \) is obtained from Eq. (4) and the solution of the Friedman-Chen nonlinear gyrokinetic equation.\(^{24} \)
\[
\left( \frac{\partial}{\partial t} + v_i \nabla_i + n_i \cdot \nabla_i \right) \delta g
\]
\[
= - \left( \frac{e}{m} \frac{\partial}{\partial t} \langle \delta L_g \rangle - \frac{c}{B_0} b \times \nabla \langle \delta L_g \rangle \cdot \nabla \delta F_0 \right) \nabla \cdot \delta \nabla F_0
\]
\[
= - \frac{c}{B_0} b \times \nabla \langle \delta L_g \rangle \cdot \nabla \delta g .
\]

(9)

Here, \( n_i \) is the magnetic drift velocity, \( \langle \cdots \rangle \) denotes gyro-phase averaging,
\[
\langle \delta L_g \rangle = \left( \delta \Phi_n - \frac{e}{c} \delta A \right) = J_0(\lambda) \left( \delta \phi - \frac{e}{c} \delta A_i \right) .
\]

(10)

\[
\delta W = \frac{2\pi^2 e^2}{|c|} \sum_{n \in \mathbb{Z}} \left| k_{[n]}(d \psi / dr) \right|^2 \frac{k_{[n]}(d \psi / dr)}{|s|} \int_{r_{p_{s,0}}}^{1} \left( \delta \Phi_{n}^{+} \delta \Phi_{n}^{-} \right) \left( \delta W_{nf} + \delta W_{nk} \right) .
\]

(11)

\[
= \lim_{v_i \to -\infty} \int_{r_{p_{s,0}}}^{1} d \psi / dr \frac{2}{|c|} \int_{r_{p_{s,0}}}^{1} \frac{\partial}{\partial \psi} \frac{C_{0}}{C_{1}} \frac{B_{0}}{B_{0}} \left( \delta F_{n}^{+} \delta F_{n}^{-} \right) \left( \delta W_{nf} + \delta W_{nk} \right) .
\]

(12)

Here, following Ref. 2, the expression of \( \delta W_{nf} \) is obtained from Eq. (12) using the “fluid” limit for the gyrokinetic particle response \( \delta g \), while \( \delta W_{nk} \) accounts for the remaining “kinetic” particle response (cf. Secs. III–VI).1

When magnetic shear vanishes at one isolated singular layer \( s = 0 \) at \( r = r_0 \) where \( k_{[n]} = k_{[0]} \), it is possible to construct the extension of Eq. (1) that, for \( |A_n| \ll 1 \), becomes8,26

\[
i S \left( \Lambda_n^2 - k_{[0]}^2 L_0 \right)^{1/2} \left( 1/n \right)^{1/2}
\]
\[
\times \left[ k_{[n]} L_0 - i \left( \Lambda_n^2 - k_{[n]}^2 L_0 \right)^{1/2} \right] = \delta W_{nf} + \delta W_{nk} ,
\]

(13)

where the connection length \( L_0 \simeq q R_0 \) in a high aspect ratio tokamak \( R_0 a \gg 1 \) with circular cross section, \( R_0 \) and \( a \) are the plasma major and minor radii, respectively; and

\[
S^2 = \frac{1}{4} q''(r_0) / q(r_0) .
\]

(14)

For small but finite magnetic shear values, one can still adopt Eq. (1) rather than Eq. (13) as long as the local expansion of \( q = q_0 + s r + 0.5 q_0 S^2 r^2 + \cdots \) can be truncated at first order in \( x \equiv (r - r_0) / r_0 \) across the kinetic/singular layer of

\[
\delta W_{nf} + \delta W_{nk} = \left( \delta \Phi_{n}^{+} \delta \Phi_{n}^{-} \right)^{-1} \left[ \frac{1}{2} \int_{-\infty}^{+\infty} \left( \frac{\partial}{\partial \theta} \delta \Phi_{n}^{+} \right) \left( \frac{\partial}{\partial \theta} \delta \Phi_{n}^{-} \right) + \frac{\delta \Phi_{n}^{+}}{\alpha} \delta \Phi_{n}^{+} + \frac{\delta \Phi_{n}^{-}}{\alpha} \delta \Phi_{n}^{-} \right] \delta F_{n}
\]
\[
+ \left( \frac{4\pi}{c} k_{[n]}^2 \right)^{\frac{1}{2}} b \times k \cdot \nabla \left\{ m \left( \frac{1}{2} \delta \Phi_{n}^{+} \right) \right\} .
\]

(15)

\[
\delta \Phi_{n}^{+} = \frac{4\pi}{c} k_{[n]}^2 \frac{B_{0}}{B_{0}} b \times k \cdot \nabla \left\{ m \left( \frac{1}{2} \delta \Phi_{n}^{+} \right) \right\} .
\]

(16)

\[ F_0 \] is the equilibrium guiding-center distribution function, and \( \mathcal{E} = v^2 / 2 \) is the energy per unit mass.

Given \( \delta \Phi_{n}^{+}(\phi) \) as solution of Eq. (6), the inertial layer response in Eq. (1) is generally obtained as

\[
\delta \Phi_{n}^{+}(\phi) = \int \left( \frac{\delta \Phi_{n}^{+}(\phi) \delta \Phi_{n}^{-}(\phi)}{\alpha} \right) \delta F_{n} \frac{1}{\alpha} \left[ \frac{\delta \Phi_{n}^{+}(\phi) \delta \Phi_{n}^{-}(\phi)}{\alpha} \right] \delta F_{n} ,
\]

(11)

where \( \delta \Phi_{n}^{+} = \lim_{\theta \to 0} \delta \Phi_{n}^{+}(\theta), \) the last term on the right hand side indicates the jump of quantity in square parentheses across \( \theta = 0 \) and \( \delta \Phi_{n}^{+} \) is the adjoint of \( \delta \Phi_{n}^{+}(\phi) \) (cf., e.g., Ref. 25). Meanwhile, the regular ideal regions response can be written as1.2

Here, \( q''(r_0) \) is the magnetic drift velocity, \( \langle \cdots \rangle \) denotes gyro-phase averaging,
Here, \( A_{n}, \delta \dot{W}_{n} \), and other physical quantities are dependent on \( r \), due to the global equilibrium profile variations. For very localized modes, whose radial envelope variation \( A_{n}(r) \) on meso-scales can be ignored, a direct comparison of Eqs. (1) and (15) yields \( \delta \dot{W}_{n} = |\delta \dot{W}_{n} | \), and the GFLDR becomes a local dispersion relation. More generally,\(^{1,10}\) Eq. (15) can be cast as an initial value problem, using the fact that \( \omega = \omega_{0} + i \delta \omega \), with \( \omega_{0} \) the typical (linear) mode frequency and \( |\delta \omega| \ll 1 \).\(^{10}\) In fact, we can describe the spatiotemporal evolution of SAW wave packets in toroidal plasmas expanding the solutions of Eq. (15) about the characteristics

\[
D_{n}(r, \theta_{10}(r), \omega_{0}) = 0. 
\]

Then, letting \( A_{n}(r) = \exp(-i \omega_{0} t) A_{n0}(r, t) \), with \( \partial \dot{A}_{n0}(r, t) \sim \gamma_{J} A_{n0}(r, t) \) and \( \gamma_{J} \) the linear growth rate, the initial value problem for \( A_{n0}(r, t) \) becomes\(^{1,23}\)

\[
\frac{\partial \dot{D}_{n}}{\partial \omega_{0}} \left( i \frac{\partial}{\partial t} \right) A_{n0} + \frac{\partial \dot{D}_{n}}{\partial \theta_{0}} \left( -i \frac{\partial}{\partial r} \theta_{0} \right) A_{n0} + \frac{1}{2} \frac{\partial^{2} \dot{D}_{n}}{\partial \theta_{0}^{2}} \left( -i \frac{\partial}{\partial r} \theta_{0} \right)^{2} A_{n0} = S_{n}(r, t). 
\]

The \( S_{n}(r, t) \) term on the right hand side represents a generic source term, including external forcing and/or nonlinear interactions.\(^{1,10,23}\)

In Secs. III–VII, we adopt the GFLDR theoretical framework and apply it to SAW/DAW excitations by wave-EP resonant interactions in fusion plasmas, which can be generally written as

\[
\omega = n \tilde{\omega}_{d} + \ell \omega_{b},
\]

for magnetically trapped particles, while, for circulating particles,

\[
\omega = n \tilde{\omega}_{d} + \ell \omega_{b} + (n \tilde{q}(r) - m) \omega_{b}.
\]

Here, \( \ell \) is the bounce/transit harmonic, \( m \) is the poloidal mode number, \( \omega_{b} \) it the bounce/transit frequency for trapped/circulating particles

\[
\omega_{b} = 2 \pi \left( \frac{q d\theta/\theta}{2} \right)^{-1},
\]

with the integral taken along a complete equilibrium particle orbit; and \( \tilde{\omega}_{d} \) is the toroidal precession frequency

\[
\tilde{\omega}_{d} = (2 \pi)^{-1} \omega_{b} \int \frac{\zeta - q \hat{\theta}}{d\theta/\theta}. 
\]

Furthermore,\(^{10,30,31}\) \( \tilde{q} \equiv \frac{q}{d\theta/\theta} \) and \( \tilde{\rho} \equiv (2 \pi)^{-1} \omega_{b} \times \frac{q}{d\theta/\theta}. \)

III. THE FISHBONE MODE

The observation of “fishbone” oscillations in the Poloidal Divertor Experiment (PDX) tokamak\(^{32}\) was the first and still remains one of the most striking evidence of resonant excitations of MHD and Alfvénic modes by EPs, with clear evidence of secular loss of supra-thermal particles\(^{33}\) and consequent drop in the fusion reactivity, made evident by a corresponding drop in the neutron signal.\(^{34}\) The theoretical interpretation of experimental observation is given in Ref. 2, where the GFLDR in the form of Eq. (1) was given for the first time. In this case, an internal kink mode with \( n = 1 \) is resonantly excited by precession resonance with trapped supra-thermal particles. Fishbone oscillations driven by transit resonance were observed shortly afterwards\(^{34}\) and later explained theoretically.\(^{28,29}\) For precession resonance, i.e., the resonance of Eq. (19) with \( n = 1 \) and \( \ell = 0 \), the mode dispersion relation reads\(^{23,34}\)

\[
i \xi s \left[ \frac{\omega}{\omega_{\Lambda}} (1 + \Delta) \right]^{1/2} = \delta \dot{W}_{f} + \delta \dot{W}_{k},
\]

where \( \omega_{\Lambda} \) is defined in Eq. (8), \( \omega_{\Lambda} = \omega(q R_{0}) \) is the usual Alfvén frequency normalization in high aspect-ratio tokamaks with circular cross section, \( r_{s} \) is the radius of the magnetic surface, and \( \Delta \propto q^{2} \) is the enhancement of plasma inertia due to geodesic curvature.\(^{35,36}\) The general derivation of \( \Lambda_{\ell}^{2} = \omega(\omega - \omega_{\Lambda})/(1 + \Delta) / \omega_{\Lambda}^{2} \) will be discussed in Sec. IV. Meanwhile, the expression of \( \delta \dot{W}_{f} \), in its simplest form, is given by\(^{37}\)

\[
\delta \dot{W}_{f} = 3 \pi \Delta q_{0} \left( 144 / 13 - \beta_{p}^{2} \right) (r_{s}/R_{0})_{1}^{2},
\]

with \( \beta_{p} = -\left( R_{0}/r_{s}^{2} \right) \int d\beta / dr d\theta / \theta \) the ratio of kinetic to magnetic pressure; and \( \Delta q_{0} = 1 - q(r = 0) \). The fluid term, \( \delta \dot{W}_{f} \), includes the contribution of the EP adiabatic and convective responses as well,\(^{28,29}\) which have been further separated from the kinetic particle response by letting

\[
\delta \xi \equiv \delta K + i \frac{e}{m} \mathbf{Q} \dot{\mathbf{F}}_{0} \mathbf{F}^{-1} (\delta \mathbf{\psi});
\]

with the operator \( \mathbf{Q} \dot{\mathbf{F}}_{0} \) defined as\(^{2}\)

\[
i \dot{\mathbf{F}}_{0} = -\frac{\partial \mathbf{F}_{0}}{\partial \mathbf{E}} \frac{\partial}{\partial t} + b \times \nabla \mathbf{F}_{0} \cdot \mathbf{V}. \]

The subdivision of the particle response as in Eq. (25) is particularly convenient in linear analyses,\(^{2,11}\) since \( \delta K \rightarrow 0 \) in the fluid ion, \( \omega \gg \tilde{\omega}_{d}, \omega_{b} \) and massless electron, \( \omega \gg \omega \), \( \tilde{\omega}_{d} \) limits, and the linearized gyrokinetic equation, Eq. (9), assuming high aspect-ratio tokamak plasma equilibria, can be cast as

\[
\delta K = \int \frac{e}{m} \left[ \mathbf{Q} \dot{\mathbf{F}}_{0} (\delta \mathbf{\psi}) - \delta \mathbf{\psi} \right] \left[ \mathbf{v}_{\perp} \cdot \mathbf{V} \right]_{\perp} \delta K,
\]

When considering short wavelength modes, \( |k_{\perp} a| > 1 \), operators

\[
\mathbf{v}_{\perp} \cdot \mathbf{V} \equiv i \omega_{d},
\]

and \( \mathbf{Q} \dot{\mathbf{F}}_{0} \) in Eq. (27) are commuting. Neglecting finite orbit width effects, assuming \( \delta \mathbf{\psi} = \delta \mathbf{\psi}_{0}(r) \exp(-i \mathbf{\gamma} \cdot \mathbf{r} - i \theta) \) and using \( \omega_{b} \gg \omega_{d} \), it is possible to obtain

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for the kinetic supra-thermal particle response,\(^2\) with \(\langle \ldots \rangle = \left(\frac{\langle f \rangle}{f} \right) = \frac{1}{\langle \ldots \rangle} \langle f \rangle \langle \ldots \rangle \rangle \) denoting bounce-averaging along the particle orbit in the equilibrium \(B_0\). Meanwhile, for a high aspect-ratio circular tokamak equilibrium, \(\delta W_k\) is given by\(^2\)

\[
\delta W_k = 4 \pi^2 \frac{\tau_0^2}{B_0^2} \int_0^\tau_0 f f d \tau \int_0^\tau_0 r d r \int \xi d \xi d \lambda \times \sum \frac{\omega \partial_0 \omega_{\theta} \omega_{\phi} \omega_{\phi} \omega_{\theta} + \omega_0 \omega_{\theta} \omega_{\phi} \omega_{\phi} \omega_{\theta}}{\omega_0 \omega_\theta \omega_{\phi} \omega_{\phi} \omega_{\theta}} \delta \phi_{\theta} \langle r \rangle d \phi_{\theta} d \phi_{\phi} d \phi_{\theta} d \phi_{\phi} d \phi_{\theta} d \phi_{\phi},
\]

where \(\delta = \mu B_0 / \xi, \delta_b = \pi / \omega_0\) for trapped particles and \(Q \Phi_0\) is computed on the mode structure \(\sim \exp(-i \omega t + \xi^2 - i \theta)\). It is worthwhile recalling\(^2\) that Eq. (23), with \(\delta W_k\) given in Eq. (30), reproduces the well-known forms of kinetic MHD energy principles in the collisionless\(^1\),\(^4\),\(^5\) and low-frequency limit\(^17\)–\(^19\) (cf. also Ref. 1). Furthermore, Eq. (30) can be extended to include the bounce-averaged response of circulating particles, for which \(\delta \Phi = \delta \Phi = \tau \omega / \omega_0\), provided that the \(\ell = 0\) resonance condition of Eq. (20) is taken into account, as recently emphasized by Merle et al.\(^38\)

As mentioned in Ref. 1, Eq. (23) admits two branches as solution: an EPM branch, the precessional fishbone,\(^2\) and a gap mode branch, the diamagnetic fishbone.\(^24\),\(^39\) The former branch satisfies\(^4\)

\[
\delta W_{pf} + \text{Re} \delta W_{nk} = 0, \quad \Rightarrow \text{determines } \omega_0, \quad (31)
\]

\[
\gamma_L \omega_0 = -\frac{\omega_0 (1+\Delta)}{\omega_0}, \quad \Rightarrow \text{determines } \gamma_L,
\]

noting that \(\delta W_f\) is independent of \(\omega_0\) in this case (see also Sec. IV); while for the latter branch, given causality constraints,\(^1\) Eq. (23) reads

\[
-3 \left[ \omega_0 (\omega_{\phi} - \omega) \right]^{1/2} = \delta W_f + \delta W_k. \quad (32)
\]

Since the early observation of fishbones\(^32\) and their consequence on plasma fusion performance,\(^32\),\(^33\) the internal kink/fishbone problem has been one of the most widely studied in the magnetic fusion literature. Further interest in the kink/fishbone stability was triggered by the observation in the Joint European Torus (JET)\(^35\),\(^42\) of sawtooth activity stabilization in plasma discharges with additional heating.\(^3\) This was explained with the strong stabilizing effect of a supra-thermal trapped particle population on the internal kink mode\(^43\),\(^44\) and confirmed by later works.\(^46\)–\(^49\) The stabilization of the internal kink mode by EPs, in the limit \(\omega_{\phi} \to 0\), was noted first by White et al.\(^50\); later, it became evident that there is a regime free of both kink and fishbone modes,\(^46\)–\(^49\) with important consequence on tokamak operations (cf. Sec. VII for further details).

As anticipated in Ref. 1, various extensions of Eq. (23) are possible, either in the description of the generalized inertia term \(\Lambda_{\phi}\) defined in Eq. (11), or in the calculation of \(\delta W_f\) and \(\delta W_k\). For \(\Lambda_{\phi}\), effects due to resistivity,\(^51\),\(^52\) diamagnetic drift,\(^53\),\(^54\) ion viscosity,\(^55\) and finite electron inertia,\(^56\),\(^57\) as well as finite Lamor radius,\(^58\),\(^59\) and Hall terms,\(^60\) have been included. A reasonably accurate, although not very recent, theoretical review can be found in Ref. 61 (see also Refs. 110, 283, and 284). For \(\delta W_f\) and \(\delta W_k\), it is worthwhile mentioning the stabilization effect due to perpendicular compressibility, first noted in Refs. 15, 62, and 63 and explicitly computed in Refs. 64–68. All these works, in their essence, are detailed analyses of Eq. (23), putting emphasis on the different spatiotemporal scales that determine the physics of the kink/fishbone branch and showing the usefulness of the GFLDR framework for theoretical analyses as well as interpretation of numerical simulation results and experimental observations (cf. Sec. VII).

Among the various applications of Eq. (23), one that has recently attracted significant theoretical as well as experimental interest is the study of the so-called “electron fishbone” (e-fishbone), where the supra-thermal particle population, which resonantly excites the internal kink mode, is due to electrons accelerated by auxiliary heating and/or current drive systems, rather than ions. Experimental observations of e-fishbones were first reported in DIII-D\(^69\) in conjunction with Electron Cyclotron Resonance Heating (ECRH) on the high magnetic field side,\(^70\) followed by further evidence in plasmas with both ECRH and/or Lower Hybrid Heating (LHH) and current drive (LHCD) (cf. Sec. VII). E-fishbones are of particular interest, for the perpendicular (bounce-averaged) trapped supra-thermal electron transport resembles that of alpha particles produced by fusion reactions in reactor relevant plasma conditions,\(^8\),\(^71\) which are both characterized by small magnetic drift orbit widths normalized to the macroscopic system size, unlike supra-thermal ions in present day experiments.\(^72\)–\(^74\) Equations (23) and (32) are readily usable for the investigation of precessional (EPM) and diamagnetic (gap mode or AE) e-fishbone branches in monotonic \(q\) profile plasma equilibria. However, due to the important role of LHH and LHCD in some experimental conditions yielding e-fishbones,\(^75\),\(^76\) the relevant fishbone dispersion relation for a hollow \(q\) profile, with the minimum value of \(q(r_f) \approx 1\) at \(r = r_f\), is obtained from Eq. (13) with \(n = 1\) and \(k_{\phi} = \Delta q_{\phi} = q(r_f) - 1\). Meanwhile, the analogue of Eq. (32) for the gap mode branch with \(s = 0,62\) is

\[
-\frac{1}{3} \left( \Delta q_{\phi}^2 - \Lambda_\phi^2 \right)^{3/4} \left[ (1 + \Delta q_{\phi}^2) / \sqrt{\Delta q_{\phi}^2 - \Lambda_\phi^2} \right] \]

\[
\delta W_f + \delta W_k, \quad (33)
\]

with \(\Lambda_\phi^2 \approx \omega_0 (\omega_\phi - \omega_{\phi}) (1 + \Delta) / \omega_\phi^2\) as noted above (cf. also Sec. IV).

It is worthwhile commenting on the \(\Delta \propto q^2\) enhancement of plasma inertia due to geodesic curvature\(^35\),\(^36\) in Eq. (23). In the “banana regime,”\(^77\) and for circular, high aspect-ratio, tokamak plasma equilibria, the correct form was first pointed out by Graves et al.\(^36\)

\[
\Delta = \left( 1.6 R_0 / r_f \right)^{1/2} + 0.5 \left( q^2 \right), \quad (34)
\]

where the \(1.6 R_0 / r_f \) factor comes from trapped and barely circulating particles; the \(0.5 q^2\) term, meanwhile, is due to well circulating particles, i.e., those particles for which the
v_i modulation along the periodic transit orbit is of order r_f R_0. It differs from the well known q^2 factor due to the intrinsic limitation of the ideal MHD model in assuming an isotropic pressure response: 2q^2 would be the result for \delta P_\perp = \delta P_||. while \delta P_\perp \neq \delta P_|| for the geodesic curvature dynamics in toroidal systems. The problem of the kinetic “bulk ion inertia enhancement” for low frequency (banana-regime) MHD modes was analyzed in Refs. 79–81, where estimates were given for both inertia enhancement as well as ion Landau damping. A more systematic analytic approach was given in Refs. 36 and 82. More recently, it was pointed out that ion Landau damping due to the precessional resonance with thermal (bulk) ions may also play a crucial role in the internal kink mode stability of ITER (see Sec. VII). Here, it is worthwhile noting that the inertia enhancement factor is identical to the zonal flow (ZF) polarizability induced by Ion Temperature Gradient (ITG) turbulence. This is not a coincidence and is due to the fact that, at long wavelengths, the gyrokinetic vorticity equation predicts that SAW compressibility due to geodesic curvature coupling at k_i = 0 is identical to the corresponding dynamics of electrostatic waves with k_i = k_\perp = 0, provided that diamagnetic effects are neglected and \omega_n < |\omega| < \omega_{pe}. For this reason, we naturally expect a correspondence between ZF polarizability and SAW inertia enhancement in the banana regime, as in Eq. (34).

In high-\beta plasmas, typical of spherical tori, the usual kink/fishbone branch may be stabilized by the reversal of the direction of the toroidal particle precession drift, as shown theoretically and observed experimentally in NSTX \cite{91} and START. However, when the distribution function of supra-thermal particles is stable with respect to the kink/fishbone excitation at the precession resonance, other precession-bounce resonances with \epsilon \neq 0 in Eq. (19) may effectively drive this mode at higher frequencies, as demonstrated by Fredrickson et al. \cite{91}. Parallel and perpendicular plasma compressions are also relevant for the accurate evaluation of the enhancement of plasma inertia when |\omega| \geq \omega_{pe}. The importance of accounting for thermal ion transit resonance in the study of resistive MHD modes was proposed in Ref. 93, while its effect on the SAW continuous spectrum was given in Refs. 94 and 95. Using the expression of \Lambda^2 computed for |\omega_{bi}| < |\omega| < \omega_{pe} \cite{94} (see Sec. IV for more details), it was demonstrated that a high frequency kink/fishbone can be excited, satisfying the same dispersion relation, Eq. (1), but with constraints. \cite{1}

High frequency fishbones have been observed in JET \cite{100} and fluctuations with similar features, consistent with Eqs. (1) and (35), were observed for the first time during D-T experiments in TFTR \cite{101,102} as recently reported. \cite{103} JET observations are consistent with theoretical predictions. \cite{71,104} Meanwhile, high frequency fishbones have been investigated numerically, showing the crucial importance to properly account for accurate mode structures, which may be significantly different from that of the usual fishbone. \cite{105}

The GFLDR theoretical framework, in summary, suggests that kink/fishbone stability can be strongly affected by kinetic effects: while potential energy is modified by supra-thermal particles as well as thermal electrons and ions, the generalized inertia is mostly affected by thermal ions, although supra-thermal ions may contribute significantly as well (see Secs. IV and V for more details). This can be noted from Eqs. (23), (32), and (33), when using the expression of \Lambda^2 obtained from Eq. (11) including supra-thermal ion response in the long wavelength limit, rather than taking \Lambda^2 \approx \omega/(\omega - \omega_{pe})(1 + \Delta)/\omega^2_A.

IV. THE LOW FREQUENCY SHEAR ALFVÉN WAVE SPECTRUM

Early investigations of the low frequency SAW/DAW spectrum in toroidal geometries were focused on kinetic descriptions of wave-particle interactions with the thermal plasma component. \cite{90,95} Theoretical studies of collisionless kinetic ballooning modes (KBMs) \cite{100,106} in particular attracted significant attention, with emphasis on the possibility of exciting short wavelength modes by EPs. \cite{4,11,109,114} The experimental observation of BAEs \cite{98,99} excited by EPs in the low-frequency SAW continuous spectrum gap due to finite thermal plasma compressibility, \cite{99,115,116} has revived the interest in this frequency range because of the impact of these fluctuations on EP confinement (see Sec. VII).

All these modes existing near the low-frequency kinetic thermal ion (KTI) gap \cite{6} are well described within the theoretical framework of the GFLDR. The generalized inertia term, Eq. (11), can be computed from the solution of the vorticity equation, Eq. (6), along with the \kappa_\perp \gg 1 limit of the linearized quasineutrality condition, \cite{0} which reads

\begin{equation}
\left( 1 + \frac{1}{\tau} \right) \left( \delta \Psi_n - \delta \Psi_n \right) + \left( 1 - \frac{\omega_{pe}}{\omega} \right) k_i^2 \beta_i^2 k_x^2 \delta \Psi_n = \frac{T_e}{\rho_0 e} \kappa_\perp \left( 1 - k_i^2 \frac{\mu B_0}{2 e} k_x^2 \right) \delta K_{\perp} \right\}, \quad (36)
\end{equation}

Here, we have expressed the non-adiabatic thermal ion response \delta g_n in terms of \delta K_{\perp}, according to Eq. (25), and adopted the small thermal ion Larmor radius ordering, consistent with Eq. (6). Furthermore, one single species of core-plasma ions with unit electric charge e has been assumed, \rho_0 is the equilibrium core plasma density, \tau = T_e/\tau, and EPs are neglected in the inertial layer, assuming that EPorbits are larger than the layer width and/or that their density is much smaller than that of the core plasma component.

The \kappa_\perp \gg 1 limits of Eqs. (6) and (36) demonstrate why thermal ions dominate the kinetic layer response and the
expression of $\Lambda_n$. Only trapped thermal and supra-thermal electrons enter via their bounce-average responses. Meanwhile, due to the structure of the “ballooning-interchange” term in Eq. (6), the bounce average electron response in the layer is smaller than that in the ideal region by the ratio of the layer width to the poloidal wavelength. Finally, the electron bounce-average response in Eq. (36) is also small. In fact, the supra-thermal component, if it exists, has small density and, thus, weakly affects the nearly constant (d.c.) component of the parallel electric field.

Meanwhile, the thermal electron component can modify the d.c. parallel electric field only for frequencies comparable with the thermal electron precession frequency,\textsuperscript{119} while typically $|\omega| \gg |n\omega_{de}|$ for SAW/DAW.

As noted in Sec. I, the GFLLDR can be computed numerically and/or analytically with various levels of approximation. Here, we discuss analytic derivations using the so called $(s, x)$ model equilibrium\textsuperscript{27} for the local description of a high aspect-ratio tokamak with shifted circular magnetic surfaces.

Equation (39) shows that mode structures can be written as

$$\phi_n = \begin{cases} \phi \sin \theta, & n \geq 0 \\ \phi \cos \theta, & n < 0 \end{cases}$$

where $\phi(\theta, \theta_k) = [\phi(\theta - \theta_k) - x \sin \theta] \sin \theta + \cos \theta$, the subscript $E$ stands for EPs, having expressed $\delta \tilde{E}_n$ in terms of $\delta \hat{K}_n$ for the core plasma components and neglected kinetic thermal electron effects, $\propto \delta \hat{K}_n$, for $|\omega| \gg |n\omega_{de}|$.\textsuperscript{119} Here, $\omega_A = v_A(\sigma q_R)$, $\omega_A$ refers to the core plasma components only, while $\omega$ includes EPs as well. In the $k^2_B \rho^2_L \to 0$ limit, and without kinetic thermal ion compression terms, $\propto \delta \hat{K}_n$, Eq. (39) reduces to the form used in analyses of KBM resonant excitations by EPs.\textsuperscript{34,114}

The generalized inertia term $\Lambda_n$ can be computed from Eqs. (36) and (39) for $k^2_B \rho^2_L \to 0$, $|\delta \tilde{E}_n| \to 0$ and $s^2 \theta^2 \to \infty$. Here, we follow Ref. 96 and consider finite mode number\textsuperscript{104,123} in order to be able to use Eq. (1) also for moderate (poloidal, toroidal) mode numbers $(m, n)$. We also consider mode structures that may have a kinetic singular (inertial) layer located away from a mode rational surface, so that $k_B q_R = (m - n)$ is generally non vanishing but still $|k_B q_R| \ll 1$. This allows us to derive $\Lambda_n$ expressions that apply for $s = 0$ as well and can be used in Eq. (13). Equation (39) shows that mode structures can be written as $\Lambda_n$.

Magnetic shear $s$ is defined in Eq. (2), $z$ is the dimensionless “ballooning” pressure gradient parameter

$$z = -R_0 q^2 d\beta/dr,$$

and $\beta$ is the ratio of kinetic to magnetic pressure, $\beta = 8\pi P_0/B_0$. In this case, Boozer coordinates\textsuperscript{120,121} (cf. Ref. 1 for more details) are easily constructed (cf., e.g., Ref. 122), with $\mathcal{J} = q R_0 (B_0/B^2)$ and, from Eq. (7)

$$\kappa^2 = [\phi(\theta - \theta_k) - x \sin \theta] \sin \theta + 1 - 2l/r + \Lambda' \cos \theta,$$
where \( \omega_{ni} = (2T)_{1/2}/(qR_0) \), \( \omega_{b(m)}^2 / \omega_{ni} = [1 + (nq - m)] / qR_0 \) and \( F(x) \) and \( G(x) \) are defined as

\[
F(x) = x(x^2 + 3/2) + (x^4 + x^2 + 1/2)Z(x),
\]
\[
G(x) = x(x^2 + 3/2) + (x^4 + x^2 + 1/2 + 3/4)Z(x);
\]  

(42)

and contain the plasma dispersion function \( Z(x) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} / (y - x) dy \). Meanwhile, the functions \( N_m(x) \) and \( D_m(x) \), to be computed at the poloidal mode number \( m \), are also polynomials containing the plasma dispersion function \( Z(x) \).

\[
N_m(x) = \left(1 - \frac{\omega_{m}}{\omega}\right) \left[x + (1/2 + x^2)Z(x)\right] - \frac{\omega_{m} \omega_{Ti}}{\omega} \left[x(1/2 + x^2) + (1/4 + x^4)Z(x)\right],
\]
\[
D_m(x) = \frac{1}{x} \left[1 + \frac{1}{x} \right] + \frac{1}{x(1/2 + x^2)}Z(x) - \frac{\omega_{m} \omega_{Ti}}{\omega} \left[x + (x^2 - 1/2)Z(x)\right].
\]  

(43)

With the typical long wavelength ordering \( |k_\perp \rho_i| \sim |\omega^2 / \omega_{b(m)}^2| \), we have \( \delta \Phi_n \sim |\omega / \omega_{b(m)}| \delta \Phi_n^0 \); i.e., modes with typical SAW/DAW polarization that characterize most unstable fluctuation structures in the KT1 frequency range from KMB to BAE modes.\(^6\) However, for \( D_m(n) \sim |\omega / \omega_{b(m)}| \ll 1 \), we have \( \delta \Phi_n \sim \delta \Phi_n^0 \), and mode structures have a mixed Alfvénic and acoustic polarization, which is that of Beta induced Alfvénic Acoustic Eigenmodes (BAAE).\(^126-128\) When this happens, \( D_m(n) \sim 0 \) gives the lowest order mode dispersion relation, and BAAE is essentially a strongly damped drift wave (DW).\(^125\) In fact, both SAW/DAW and BAAE branches are characterized by a predominantly sinusoidal (a.c.) parallel electric field perturbation. However, there are two fundamental differences between former and latter kind of fluctuations: (i) \( \delta E \parallel \) is an important component of the BAAE mode structure, while it is a perturbation for long wavelength SAW/DAW modes; (ii) typical BAAE frequencies are lower than those of SAW/DAW, so that wave-particle interactions with thermal ions are stronger. These are, also, the fundamental reasons why Landau damping is generally much stronger for BAAE than for SAW/DAW, as shown in Refs. \( 124 \) and \( 125 \), unless sound wave frequency and \( \omega_{ni} \) are well separate; i.e., \( T_e \gg T_i \). Another possible condition for reducing the otherwise generally strong BAAE Landau damping occurs for sufficiently short wavelengths such that DW is near its instability threshold\(^24,128\) (cf. also Sec. VII). In these conditions, however, BAAE and KMB are typically coupled due to the strength of diamagnetic effects; and polarization of both fluctuation branches is significantly modified and much closer to that typical of Alfvénic fluctuations.\(^124\)

It is also worthwhile reminding that, for sufficiently low frequencies comparable to the thermal ion bounce frequency, \( \omega_{ni} \), kinetic responses of magnetically trapped core plasma ions must be accounted for,\(^119,123\) especially when realistic comparisons with experimental observations are made.\(^123,129-131\) (cf. Sec. VII). In this case, Eq. (40) can either be evaluated numerically or Eq. (41) can be extended to account for magnetically deeply trapped thermal ions. The resulting \( \Lambda_n^2 \) expression (cf. Refs. \( 119 \) and \( 124 \) for detailed discussions) smoothly connects to \( \Lambda_n^2 = \omega(\omega - \omega_{ni})/(1 + \Delta)^2 / \omega_{b(m)}^2 \) for \( |nq - m| \ll |\omega| \ll \omega_{ni} \), with \( \Delta \) given by Eq. (34); and reduces to Eq. (41) for \( |\omega| \gg \omega_{ni} \), with high mode number and \( |nq - m| \to 0 \). For high mode number and \( |nq - m| \to 0 \) such that \( \omega_{b(m)}^2 = \omega_{ni} \), Eq. (41) also reduces to the expression derived in Ref. \( 96 \)

\[
\Lambda_n^2 = \frac{\omega^2}{\omega_{b(m)}^2} \left(1 - \frac{\omega_{m}}{\omega}\right) + \frac{\omega_{m} \omega_{Ti}}{\omega} \left[1 - \frac{\omega_{m}}{\omega}\right] F(\omega / \omega_{ni})
\]
\[
- \frac{\omega_{m} \omega_{Ti}}{\omega} G(\omega / \omega_{ni}) - \frac{N_m^2(\omega / \omega_{ni})}{D_m(\omega / \omega_{ni})},
\]  

(44)

which, in the \( |\omega| \gg \omega_{ni} \) limit, yields Eq. (35). Setting \( \Lambda_n^2 = 0 \); i.e., looking at the accumulation points of the SAW continuous spectrum,\(^1 \) Eq. (35) demonstrates that BAE accumulation point frequency in the long wavelength limit is degenerate with the frequency of the Geodesic Acoustic Mode (GAM),\(^132\) as pointed out in Refs. \( 6 \) and \( 21 \) and noted in Refs. \( 133 \) and \( 134 \). The GAM/BAE accumulation point degeneracy follows, again, naturally from the gyrokinetic vorticity equation\(^1,86-88\) predicting that SAW compressibility due to geodesic curvature coupling at \( k_\perp = 0 \) is identical to the corresponding dynamics of electrostatic waves with \( k_\parallel = k_\perp = 0 \), provided that diamagnetic effects are neglected. The coupling of SAW/DAW, GAM, and acoustic branches has also been addressed in the investigation of drift sound waves in the W7-AS stellarator.\(^135\)

Fluctuations of SAW/DAW and BAAE branches are described by the same GFLDR, Eq. (15),\(^129\) or by its global equivalent, Eq. (1), if macroscopic modes are considered. Expressions for \( \delta W_{W} \) and \( \delta W_{k} \) can be obtained from Eq. (16), and, for the \( (s, x) \) model equilibrium,\(^27\) can be rewritten as\(^3,4\)

\[
\delta W_{W} = \left(\delta \Phi_{\delta n0}^0 \delta \Phi_{\delta n0}^0 \right)^{-1} \frac{1}{2} \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial n} \delta \Phi_{\delta n}^0 \right] d\theta,
\]
\[
\times \left(\frac{\partial}{\partial \theta} \delta \Phi_{\delta n}^0 \right) \delta \Phi_{\delta n}^0 \left(s - \cos \theta \right)^2 - \frac{\cos \theta}{k_\perp^2} \right] \delta \Phi_{\delta n}^0 \right] d\theta,
\]
\[
+ \frac{\hat{\epsilon} \mathcal{L}(\theta, \theta_{k})}{k_\perp^2} \right] \delta \Phi_{\delta n}^0 \right] d\theta,
\]
\[
\delta W_{k} = \left(\delta \Phi_{\delta n0}^0 \delta \Phi_{\delta n0}^0 \right)^{-1} \left(\frac{1}{2} \right) \int_{-\infty}^{\infty} \delta \Phi_{\delta n0}^0 \right] d\theta.
\]

(45)

Here, \( \hat{\epsilon} \mathcal{L} \) is defined as

\[
\hat{\epsilon} \mathcal{L} = - \delta \Phi_{\delta n0}^0 \left(\frac{4 \pi}{B_0^2} \right) R_0 R_0 \left[ m E \right] \left[ \mu B_0 + v_\perp^2 \right] \left[ \mu B_0 + v_\perp^2 \right] \right] d\theta.
\]

(46)

(47)
evaluated using $\delta \Psi_n^{(0)} = 1$ as trial function and neglecting $\propto \tilde{z}_F$ contributions, as EP finite orbit widths are typically dominated by magnetic drifts rather than by finite Larmor radii.\cite{FZ13,FZ14} Meanwhile, Eq. (46) can also be computed analytically, by direct substitution of approximate solutions of Eq. (27) obtained for either well circulating ($v_l \approx \text{const}$) or deeply trapped EPs ($v_l \approx q_B \omega_d \theta_b \cos \phi_{oE}$, with $\omega_o = (r/R_0)^{1/2} \varepsilon_2^{1/2}/(q_R 0)$ and $\theta_b$ the magnetic bounce angle)\cite{FZ13,FZ14} Denoting by the superscript “n” (untrapped) circulating particles and by the subscript “r” magnetically trapped particles, one can let $\delta \tilde{W}_{nk} = \delta \tilde{W}_{nk}^\prime + \delta \tilde{W}_{nk}^\prime$ and obtain\cite{FZ13,FZ14}

$$\delta \tilde{W}_{nk}^\prime = \frac{\pi^2}{2} \frac{e^2 q^2 R_0^2}{\mu c^2 n} \frac{Q}{\Omega_d(1 + \Delta_2^{3/2})} \frac{\omega - \omega_{\text{nr}}}{{\omega_0^2 - \omega^2}} \bigg|_{l_E},$$

(48)

$$\delta \tilde{W}_{nk} = \frac{\pi^2}{2} \frac{e^2 q^2 R_0^2 \delta_{nk}^{B_0} \sum_{l_E|\phi = 1}}{\mu c^2 n} \frac{d\Omega_{d0}}{k_0^2} \frac{1}{\omega_{\text{nr}} - \omega} + \frac{\theta_b}{\Delta_2^{3/2}} \frac{\omega - (\omega_{\text{nr}} - \omega_{\text{nr}}^2)}{\Delta_2^{3/2} \omega_{\text{nr}} - (\omega_{\text{nr}} - \omega_{\text{nr}}^2)} \bigg|_{l_E}. \tag{49}$$

Here, $\tilde{F}_{\text{AE}}$ is assumed to be symmetric in $v_l$, for the sake of simplicity. Meanwhile,\cite{FZ13,FZ14} $\omega_{\text{nr}} = v_l/(qR_0)$, $\tau_b = 2\pi/\omega_b$, $\omega_{\text{nr}} = \Omega_d = -\mu(B_0 + v_l k_b)/\Omega_b (Q_R 0)$ (cf. Eq. (28)), and a Padé approximation has been adopted for Bessel functions accounting for both finite Larmor radius and finite magnetic drift orbit width by means of the quantities $\Delta_2 = (k_b^2/4)(\rho_b^2 + \rho_b^2/2)$, $\Delta_4 = (k_b^2/2)(\rho_b^2 + \rho_b^2)$, $\Delta_5 = (k_b^4/4)(\rho_b^2 + \rho_b^2/2)$, with $\rho_b = 2\mu B_0 / \Omega_0^2$; $k_b^2 \rho_b^2 = \Omega_b^2 / \omega_{\text{nr}}^2$ and $k_b^2 \rho_b^2 = \theta_b^{2} \Omega_b^2 / \omega_{\text{nr}}^{2}$,\cite{FZ13,FZ14} Furthermore, only the dominant transit and bounce resonances have been considered for the sake of simplicity. Equations (48) and (49) demonstrate that the typical lower bound of $\lambda_\perp$ for SAW/DAW excited by EPs is set by the characteristic EP (magnetic drift) orbit width $\rho_E$, $\lambda_\perp \approx \rho_E$, as demonstrated in the original works by Refs. 3, 4, 136, and 137 (cf. also Sec. V).

For increasing mode numbers, Eqs. (15), (44), (45), (48), and (49) demonstrate that SAW/DAW undergo a gradual transition from a prevalent EP drive, for $\lambda_\perp \approx \rho_E$, to a prevalent core plasma drive, for $\rho_1 \ll \lambda_\perp \ll \rho_E$. This is readily noted from the large argument expansion in the plasma dispersion functions of Eq. (44), whose real part reproduces Eq. (35) and, accounting for resonant wave-particle interactions with thermal ions, yields\cite{FZ13,FZ14}

$$A_n = \frac{\omega^2}{\omega_{\text{AE}}^2} - \frac{\omega_{\text{AE}}^2}{\omega_{\text{AE}}^2} \left[ 1 + \frac{\omega_{\text{AE}}^2}{q^2 \omega^2} \times \left\{ \frac{(46/49)}{32/49} (T_s/T_i) + \frac{(79/49)}{32/49} (T_s/T_i)^2 \right\} \right] + i\sqrt{\pi q^2} \frac{e^{-\omega q^2 \omega_{\text{AE}}^2}}{\omega_{\text{AE}}^2} \left( \frac{\omega_{\text{AE}}^2 - \omega_{\text{AE}}^2}{\omega_{\text{AE}}^2 + T_s/T_i} \right). \tag{50}$$

This shows that the SAW accumulation point at $\Lambda_\perp = 0$ acquires a positive imaginary part for $\omega_{\text{TE}} > \omega_{\text{AE}}^2 / \omega_{\text{AE}}^2$, which corresponds to the excitation of an Alfvén Ion Temperature Gradient (AITG)\cite{FZ13,FZ14} driven mode when the causality constraint $\delta \tilde{W}_f + \Re e \delta \tilde{W}_E < 0$ is satisfied. Numerical analyses of AITG stability are reported, e.g., in Refs. 138–142. Thus, there exists a broad range of mode numbers in the same frequency range $[\omega \leq \Omega(10^{-1}) \omega_{\text{AE}}]$, predicted theoretically\cite{FZ13,FZ14,FZ14} and observed experimentally\cite{FZ13,FZ14} (cf. Sec. VII), where both core and energetic plasma component act as free energy source with possible important consequence for cross-scale coupling and nonlinear dynamics in burning plasmas.

For further shorter wavelengths, thermal ion finite Larmor radius and magnetic drift orbit width become important in Eqs. (36) and (39), and yield the discretization of the SAW continuous spectrum (cf. Sec. V). Equation (40) can then be generalized to\cite{FZ13,FZ14}

$$\frac{\partial^2}{\partial t^2} \delta \Psi_n^{(0)} + \Lambda_\perp^2 \omega \delta \Psi_n^{(0)} - \delta \tilde{Q}_n^{2/(\omega)}(\omega) \delta \tilde{Q}_n^{(0)} = 0, \tag{51}$$

$$\delta \tilde{Q}_n^{2/(\omega)}(\omega) = \frac{9^2 k_p^2 \omega_{\text{AE}}^2}{\omega_{\text{AE}}^2} \left[ \frac{3}{4} \left( 1 - \frac{\omega_{\text{AE}}}{\omega_{\text{AE}}} - \frac{\omega_{\text{AE}}}{\omega_{\text{AE}}} \right) \right] + q^2 \frac{\omega_{\text{AE}}}{\omega} S_{n}(\omega) + \left( \frac{\Lambda_\perp \omega_{\text{AE}}}{\omega} \right)^4 \frac{1}{\tau + (\omega_{\text{AE}}/\omega)} \right], \tag{52}$$

where the frequency dependent function $S_{n}(\omega)$ accounts for finite magnetic drift orbit width\cite{FZ13,FZ14}
Here, the functions $N, D, F, G$ are those defined in Eqs. (42) and (43), with subscripts $m \approx nq$ dropped from $N, D$ for simplicity. Meanwhile $V, W, H, I$ and $T, U, L, M$ functions are defined as

$$V(x) = x + (x^2 + 1)Z(x),$$
$$W(x) = x^3 + 2x + (x^4 + (3/2)x^2 + 3/2)Z(x),$$
$$H(x) = x^5 + 2x^3 + 3x + (x^6 + (3/2)x^4 + (3/2)x^2 + 3/4)Z(x),$$
$$I(x) = x^7 + (3/2)x^5 + (7/2)x^3 + (27/4)x + (x^8 + (9/4)x^4 + 3x^2 + 15/8)Z(x),$$
$$T(x) = x^3 + (5/2)x + (x^4 + 2x^2 + (3/2))Z(x),$$
$$U(x) = x^5 + 3x^3 + (13/2)x + (x^6 + (5/2)x^4 + (9/2)x^2 + (15/4))Z(x),$$
$$L(x) = x^7 + (5/2)x^5 + (19/4)x^3 + (63/8)x + (x^8 + 2x^6 + 3x^4 + 3x^2 + 3/2)Z(x),$$
$$M(x) = x^9 + 2x^7 + (11/2)x^5 + (25/2)x^3 + (201/8)x + (x^{10} + (3/2)x^8 + 4x^6 + (15/2)x^4 + 9x^2 + 21/4)Z(x).$$

In Eq. (53), functions without subscript are computed at $\omega/\omega_i$, while functions with subscript 1/2 are computed at $\omega/2\omega_i$ and account for wave-particle interactions at the first sideband transit resonance $2\omega_i$. Noting that for $|\omega| \gg \omega_i$, $Q_n^2/(x^2)^2k^2 \approx Q_n^2/k^2 \rightarrow \rho_0^* = \omega^2/(\omega_i^2 k^2)$, defined in Ref. 1, provided that small but finite Landau damping is maintained for thermal particles and finite resistivity is added in the parallel Ohm’s law,

$$\rho_0^* = \frac{3}{4} (1 - i \delta_0) + \frac{\omega_i}{\omega} (1 - i \delta_\infty),$$

Equation (51) is the generalization of low frequency $|\omega| \approx \omega_i k^2/4 \omega_i k \rangle$ KAW in tokamak plasmas. Note, however, that the explicit expression of $Q_n$ given in Eq. (52) does not include magnetically trapped particle effects.

Equation (51) can be readily solved, noting Eq. (11), to derive the following GFLDR extended to short wavelengths\textsuperscript{17}

$$-2Q_n^{1/2} \frac{\Gamma(3/4 - \Lambda_n^2/4Q_n)}{\Gamma(1/4 - \Lambda_n^2/4Q_n)} = \delta \dot{W}_{nf} + \delta \dot{W}_{nk},$$

where we have introduced the Euler-function. For $|\Lambda_n^2/4Q_n| > 1$, the left hand side of Eq. (56) reduces to $i \Lambda_n$; i.e., Eq. (15) is recovered. This property shows that the discrete structures (“granularity”) of the SAW continuum spectrum depend on the spatial scales $|Q_n| \sim k \rho_i$ as well as the time scales $|\Lambda_n| \approx |\omega/\omega_i|$ on which the spectrum is “observed”\textsuperscript{1,113,144}. For sufficiently long spatial and temporal scales the discretized SAW spectrum behaves nonetheless as a “true” continuum (cf. also Sec. V). Thus, Eq. (56) describes a variety of kinetic SAW/DAW fluctuations, including, e.g., BAE and kinetic BAE (KBAE).\textsuperscript{145} Equation (56) has also been derived and analyzed extensively by Nguyen et al.,\textsuperscript{146} to be then adopted in modeling of BAE observations in Tore Supra\textsuperscript{147} (cf. Sec. VII).

In tokamak plasma equilibria with hollow-$q$ profiles, characterized by $s < 0$ inside the minimum-$q$ surface, a natural extremum of the SAW continuum and minimization point for continuum damping is the magnetic surface with $s = 0$. In this case, the GFLDR form is that of Eq. (13), which describes Reversed Shear AE (RSAE)\textsuperscript{148,149} or Alfvén Cascades (AC).\textsuperscript{150,151} As the parallel wave vector at $s = 0$, $k_{\parallel 0}$, is generally finite, the RSAE/AC frequency is expected to be typically larger than that of BAEs. Thus, $\Lambda_n$ expression of Eq. (50) can be used in Eq. (13); which can be further simplified to\textsuperscript{7,152}

$$iS \left( \Lambda_n^2 - k_{\parallel 0}^2 q R_0^2 \right)^{1/2} \left( k_{\parallel 0} q R_0 / n \right)^{1/2} = \delta \dot{W}_{nf} + \delta \dot{W}_{nk}.$$  

Here, we have assumed $\Lambda_n^2 \approx k_{\parallel 0}^2 q^2 R_0^2$ and $L_0 \approx q R_0$ for modes characterized by small frequency shift with respect to the SAW accumulation point in a large aspect-ratio tokamak with shifted circular magnetic flux surfaces. Using $n > 0$ as reference, it is readily noted that the optimal conditions for exciting RSAE/AC at the minimum-$q$ surface are $-1/2 < n q_0 - m < 0$ and $1/2 < n q_0 - m + 1 < 1$. In fact, when this is verified, the $(n, m)$ SAW continuum has a maximum at $s = 0$ below the corresponding minimum of the $(n, m - 1)$ SAW continuum.\textsuperscript{150} Therefore, the continuum damping due to non-local coupling of mode structures with the SAW continuous spectrum is minimized, since $(n, m)$ and $(n, m - 1)$ SAW continua do not intersect.\textsuperscript{152} Furthermore, a wide frequency gap is formed between the local maximum of the $(n, m)$ and the local minimum of the $(n, m - 1)$ SAW continua.\textsuperscript{150,151} The low frequency RSAE/AC branch of AEs can, thus, exist marginally above the local (maximum) accumulation point of the $(n, m)$ SAW continuum, provided that\textsuperscript{1}

$$\delta \dot{W}_{nf} + \text{Re} \delta \dot{W}_{nk} > 0.$$  

For this reason, RSAEs/ACs have characteristic features that are similar to those of Global Alfvén Eigenmodes (GAEs).\textsuperscript{153–156} Equation (58) can be fulfilled either by compressibility effects of very energetic EPs with large orbits,\textsuperscript{151} yielding $\text{Re} \delta \dot{W}_{nk} > 0$,\textsuperscript{152} or by thermal plasma (toroidal) geometry\textsuperscript{157} and density gradient\textsuperscript{158} effects. In general, in the absence of EPs, Eq. (58) for weak/vanishing magnetic shear requires local macroscopic plasma stability; i.e., it reduces to
the Mercier stability criterion. This result was shown explicitly, investigating the compressibility effects on RSAE/AC in the “ideal region,” while the thermal plasma compression effect on RSAE/AC in the “inertial layer,” i.e., the $\propto \omega_{\text{BAE}}^2$ term in Eq. (50), was pointed out by Ref. 161. Equation (58) provides only a necessary but not sufficient criterion for RSAE/AC excitation, as mode drive is generally due to resonant EPs, $\Im m \delta \tilde{W}_{\text{sk}} > 0$. In present day experiments, resonant and non-resonant EPs are typically well distinct classes of particles, unlike what generally happens in burning plasmas.\textsuperscript{7,152}

The experimental feature of RSAE/AC to typically chirp upward in frequency is readily understood, as the accumulation point of the $(n, m)$ SAW continuum increases in absolute frequency for decreasing $q_0$ due to, e.g., resistive current diffusion.\textsuperscript{150,151} When the RSAE/AC frequency reaches the TAE frequency gap, $(n, m)$ and $(n, m-1)$ SAW continua intersect and toroidicity effects become important.\textsuperscript{152,157} Accurate modeling of RSAE/AC mode frequencies and growth rates has been applied to experimental observations in JET to explain evidence of upward mode frequency chirping, and eventually downward, after reaching the TAE frequency gap\textsuperscript{162} (cf. Sec. VII).

V. TOROIDAL ALFVEN EIGENMODES

After theoretical prediction and observation of GAEs\textsuperscript{153–156} that, however, are generally damped because of their coupling with the SAW continuous spectrum,\textsuperscript{163–166} TAEs\textsuperscript{22} are the first and most renowned example of nearly undamped AEIs in toroidal plasmas, theoretically predicted before their experimental observation\textsuperscript{167,168} (cf. Sec. VII) and widely studied as paradigm problem for their potential impact on EP confinement.\textsuperscript{10}

From the general structures of SAW continuous spectrum,\textsuperscript{1} one readily derives that $k_{\parallel}^2 q_0^2 R_0^2 \simeq 1/4$ and $\omega^2 \simeq \omega_{\text{TAE}}^2/4$ for TAE in circular plasmas with large aspect-ratio $R_0/a$. Following the original work of Refs. 22 and 169 and adopting the $(s, x)$ model equilibrium\textsuperscript{27} introduced in Sec. IV, Eqs. (36) and (39) are readily specialized to TAE and yield

$$
\frac{\partial^2}{\partial t^2} \delta \tilde{\Psi}_n \left[ \begin{array}{c}
\frac{\omega^2}{\omega_0^2} (1 + 2\epsilon_0 \cos \vartheta) + \left[ \frac{2 \cos \vartheta}{k_{\parallel}^2} - \frac{(s - 2 \cos \vartheta)^2}{k_{\perp}^4} - \frac{k_{\perp}^2}{4} \right] \delta \tilde{\Psi}_n \\
+ \frac{4\pi R_0 q_0^2 g(\vartheta, \theta_k)\alpha}{B_0} \left[ \left( \mu B_0 + v_\parallel^2 \right) \left( m_e \delta K_m + m_e \delta K_m \right) \right] \right] = 0. \quad (59)
\end{array} \right.
$$

Here, notations are those introduced in Sec. IV, $\epsilon_0 = 2(r/R_0 + \Delta')$, core plasma diamagnetic frequencies have been neglected compared to TAE frequency, and it is assumed that EP finite orbit widths are typically dominated by magnetic drifts rather than by finite Larmor radii\textsuperscript{3,4} [cf. Eq. (47) and following discussion]. Meanwhile, the expression of $\rho_0^2$ is given by Eq. (55); or Eq. (52) in the high-frequency limit (cf. Sec. IV), where $\delta_r = 0$ and\textsuperscript{170,171}

$$
\delta_r = \frac{\sqrt{2/3} (\nu_0/\omega)^{1/2}}{\epsilon_0/(\nu_0/\omega)^{3/2}} \left[ 1.4 + 0.25 \ln \left( 1 + \frac{\epsilon_0 \omega_0}{\nu_0} \right) \right]^{-3/2},
\quad (60)
$$

with

$$
\nu_0 = \frac{4\pi e^2 n_e \ln \Lambda}{m_e^{1/2} (2T_e)^{1/2}},
\quad (61)
$$

and $\ln \Lambda$ the Coulomb logarithm. Assuming $\delta_r = 0$ corresponds to neglecting higher order corrections to the usual Landau collisionless dissipation\textsuperscript{172} which is dominated by the $\propto \delta K_m$ term. Similarly, electron Landau damping is predominantly given by the $\propto \delta K_m$ contribution, originally dropped in Eq. (39) at low-frequency (cf. Sec. IV), while the expression of $\delta_r$ in Eq. (60) describes other dissipative effects associated, e.g., to collisions with trapped electrons.\textsuperscript{173–175} Typically, the most important TAE dissipation mechanism due to electrons is the trapped electron collisional damping.\textsuperscript{175}

Consistent with the general case, discussed in Ref. 1, Eq. (59) suggests the existence of two-scale structures for the solutions $\delta \tilde{\Psi}_n$; $\vartheta_0 \sim 1$ representing periodic variations due to toroidal geometry; and $\vartheta_1 \sim \epsilon_0^{-1} \gg 1$ characterizing the radial “singular” structures of the SAW continuous spectrum. For $|\vartheta| \sim k_{\parallel} \gg 1$, $\delta \tilde{\Psi}_n$ can generally be written as\textsuperscript{144,176}

$$
\delta \tilde{\Psi}_n^{(\pm)} = \epsilon_p^{(\pm)} \left[ A(\vartheta_1) \cos(\vartheta_0/2) \pm B(\vartheta_1) \sin(\vartheta_0/2) \right], \quad (62)
$$

where $(\pm)$ refers to the sign of $\vartheta_1$, and $\epsilon_0^{(\pm)}$ give the parity of the mode structures. Taking $\epsilon_0^{(\pm)} \equiv 1$ for reference, $\epsilon_0^{(\pm)} = \pm 1$ denotes even/odd mode structures, respectively. By direct substitution of Eq. (62) into Eq. (59), it is possible to derive the governing equations for $A(\vartheta_1)$ and $B(\vartheta_1)$.\textsuperscript{144,176}

$$
A'(\vartheta_1) = (\Gamma_+ - \frac{s^2 \nu_0^2 \rho_{e0}^2}{4} B(\vartheta_1)), \quad (63)
$$

$$
B'(\vartheta_1) = -\left( \Gamma_+ - \frac{s^2 \nu_0^2 \rho_{e0}^2}{4} \right) A(\vartheta_1).
\quad (63)
$$

Here, $\Gamma_\pm = (\omega_0^2 \rho_{e0}^2/4) (1 \pm \epsilon_0) - 1/4 - \beta_1$, with $\beta_1 = \beta_{1e} + \beta_{1E}$ and\textsuperscript{165,177–180}

$$
\beta_{1e} = \frac{\pi \rho_{e0}^2}{B_0} \sum_{j\neq 0} \sum_{n_{\text{eff}}} \sum_{l=1,3} m_l \langle \mu B_0 + v_\parallel^2 \rangle \left( \frac{\omega_0 F_0}{\omega_{\text{eff}}^2 - 4 - \omega^2} \right), \quad (64)
$$

and $\rho_{e0}$ the electron density. This leads to the eigenvalue problem

$$
\Delta(\vartheta_1) \frac{A(\vartheta_1)}{B(\vartheta_1)} = \frac{\omega_0^2 \rho_{e0}^2}{4} \frac{B(\vartheta_1)}{\omega_{\text{eff}}^2 - 4 - \omega^2}
\quad (65)
$$

which is solved in Ref. 169.
This term, which assumes for simplicity \( \tilde{F}_0 \) to be symmetric in \( \theta_0 \), describes TAE wave-particle interactions of the core plasma component at the fundamental (\( \theta_0 = \pm \theta_0 ) \) and first sideband (\( \theta_0 = \pm \theta_0 + \pi / 2 ) \) transition resonances; i.e., it accounts for electron \(^{165,177-180} \) and ion \(^{181} \) Landau damping. Note that \( \beta_1 \) and \( \rho_0^2 \) terms in Eq. (63) contribute to the generalized plasma inertia response, discussed already for MHD modes (cf. Sec. III) and low frequency SAWs (cf. Sec. IV). In the long wavelength limit, the generalized inertia can further include the EP contributions by extending \( \beta_1 \), of Eq. (64) to \( \beta_{1E} \) of the EP component. \(^{165,177-180} \) It is then possible to note that ER drive linearly increases with the toroidal mode number \( n \) until finite orbit width effects become important for \( |k_{n0}| = \kappa^{-1} \sim |\Gamma_\tau / \Gamma_\tau|^{1/2} \sim O(\epsilon_0) \) (cf. Ref. 1). This suggests that high-\( n \) modes play the dominant role in fusion plasmas, \(^{177} \) although the first numerical investigations of TAE mode structures \(^{169} \) and stability \(^{179,182-188} \) focused on low-\( n \) modes for intrinsic limitations of numerical simulation capabilities. In this long wavelength regime, it is possible to take into account small but finite EP orbit widths, which modify the structure of Eq. (63) and generalize the expression of \( \rho_0^2 \) in a qualitative and quantitative fashion that extends wave-EP resonances to the \( \theta_0 = \pm \theta_0 + \pi / 2 \) transition sideband. \(^{189,190} \)

We start investigating the GFLDR applied to long wavelength TAE in the form of Eq. (15), introducing the notation \( \Lambda_{\theta_0} \), \( \delta W_{nf} \), and \( \delta W_{nk} \) for generalized inertia response and potential energies, where the subscript \( T \) stands for the inclusion of finite toroidal coupling effects, which are crucial for TAE. \(^{22,191} \) Neglecting small but finite orbit width effects of both core and energetic plasma components, Eq. (63) yields \( A(\theta_0) \approx \langle - \Gamma \rangle^{1/2} \exp(-\langle \Gamma \rangle \theta_0) \rangle \) and \( B(\theta_0) \approx \Gamma^{1/2} \exp(-\langle \Gamma \rangle \theta_0) \rangle \), with \( \Gamma = \langle - \Gamma \rangle \approx \Gamma_\tau \). From Eq. (11), it is then readily shown that \( \Re e^\Gamma > 0 \) identifies TAE (\( \Re e^\Gamma < 0 \) identifies EPM; cf. Sec. VI)

\[
i\Lambda_{\theta_0} = (1/2)B(0)/A(0) = (1/2)(-\langle \Gamma \rangle / \langle \Gamma \rangle) \). \quad (65)
\]

Noting that the dominant kinetic interactions in the long wavelength limit occur in the kinetic layer, the GFLDR Eq. (15) is fully determined provided that \( \delta W_{nf} \) is obtained from Eq. (45), given the function \( \delta \Phi_n = \delta \Psi_n \). In general, \( \delta \Psi_n \) must be determined numerically, but, from the matching condition with the solution given in Eq. (62), it can be shown that \(^{192-194} \)

\[
\delta W_{nf} \equiv \left[ 1 - Z_f(s, \theta_0) \right] i\Lambda_{\theta_0} - \left( B(0) - A(0) \right) \frac{2A(0)^2}{2A(0)^2} \times G_f(s, \theta_0) - \left( B(0)^2 + A(0) \right) \frac{2A(0)^2}{2A(0)^2} \times H_f(s, \theta_0) \cos \theta_0 + L_f(s, \theta_0) \sin \theta_0, \quad (66)
\]

where \( Z_f, G_f, H_f, L_f, \) determined numerically, are periodic functions of \( \theta_0 \) and of parameters defining the local plasma equilibrium; i.e., in this case, \( (s, \theta_0 ) \) (cf. Sec. IV). Note that Eq. (66) is frequency dependent through the mode structure, \( \Lambda(0) \) and \( B(0) \), which depends on the mode frequency location with respect to the SAW continuum spectrum. This is in contrast with respect to the case of low frequencies, discussed in Secs. III and IV as well as Ref. 1, where \( \delta W_{nf} \) is independent of \( \omega \). In special cases, it is possible to give analytic expressions of \( Z_f, G_f, H_f, L_f, \) \(^{194} \) e.g., for \( |s|, |\theta_0| < 1 \), when Eq. (62) gives a good trial function in the whole \( \varphi \)-space \(^{22,165,177,192,195,196} \) and \( Z_\Gamma \approx 1, G_\gamma \approx |\varphi|/4, (H_\gamma \cos \theta_0 + L_\gamma \sin \theta_0) \approx (|\varphi|/4)(\theta_0 - 2k_0 \cos \theta_0), \) with \(^{192-196} \)

\[
\kappa(s) \approx \frac{1}{\delta} \left( 1 + \frac{1}{|s|} e^{-1/|s|} \right). \quad (67)
\]

Thus, Eq. (66) becomes

\[
\delta W_{nf} \approx \frac{1}{8} \frac{\varphi}{s^2} \left( 1 + 2\kappa(s) \cos \theta_0 - \frac{s}{\kappa(s)} \right) \frac{B(0)^2 - A(0)^2}{A(0)^2} \left( 1 - 2\kappa(s) \cos \theta_0 + \frac{s}{\kappa(s)} \right), \quad (68)
\]

which implies that only the even parity TAE \( (|B(0)/A(0)| < 1) \) can exist for moderate \( |s|, \theta_0 < 1 \), while the odd parity TAE \( (|B(0)/A(0)| > 1) \) is forbidden. \(^{22} \) In general, only one parallel eigenstate of TAE is identified using simplified model equilibria, \(^{22,192,195} \) while different TAE radial eigenstates exist in the number of the effectively coupled poloidal harmonics (cf. Ref. 1). The parity of the parallel eigenstate \( \delta \Phi_n \), is typically mixed and changes from even, near the lower SAW accumulation point of the TAE frequency gap, to odd, near the upper accumulation point. More general equilibria and/or the presence of a plasma free boundary in the TAE localization domain \(^{197} \) may give rise to more simultaneous TAE parallel eigenstate branches.

The TAE GFLDR in the form of Eq. (15) can be used to calculate the global dispersion relation (cf. Ref. 1) with the corresponding radial eigenstates and mode structures. \(^{192,194} \) It is then possible to compute TAE damping due to non-local coupling to the SAW continuum from the radial locations where it is resonantly excited. \(^{165,177} \) In the long wavelength limit and for TAE mode well localized inside the frequency gap, this non-local coupling is exponentially small and explains why TAEs, unlike GAEs, \(^{163-166} \) are generally nearly undamped fluctuations. Adopting the \( (s, \theta_0 ) \) model equilibrium, the corresponding TAE continuum damping \(^{198} \) has the general form

\[
\frac{\gamma_{ce}}{\omega} = \frac{1}{4} \frac{1}{\varphi^2} \left( 4 \sqrt{2} e^{-\varphi^2/4} \right)^{3/2} \left[ \Gamma(s, \theta_0) e^{-2\varphi^2/4} + \Gamma_n(s, \theta_0) e^{-2\varphi^2/4} \right]. \quad (69)
\]

Here, \( L_a^{-1} \equiv \partial_\Gamma \ln \sigma_a^2 \) and the functions \( \Gamma(s, \theta_0 ) \) and \( \Gamma_n(s, \theta_0 ) \) represent, respectively, the absorption rate at the lower and upper SAW continuum. Meanwhile, \( \Gamma(s, \theta_0 ) \) and \( \Gamma_n(s, \theta_0 ) \) are the “tunneling” factors that describe the TAE wave cut-off while propagating towards the lower and upper SAW continua. All these functions have been calculated numerically for arbitrary values of \( (s, \theta_0 ) \). \(^{192,194} \) For \( |s|, \theta_0 < 1 \), these functions have the following analytical expression: \(^{192,194,196} \)
Thus, it is possible to note that, as \( x \to \pm k_c (1 - 2k/s) \), the TAE mode merges into the lower SAW continuum accumulation point\(^{177,178,195}\) and \( \pm \omega_c / |\omega| \) from Eq. (69) diverges, due to the strengthened coupling of the mode with the continuous spectrum.\(^{196}\) Asymptotic techniques have been used to calculate high-n TAE spectra,\(^{144,176,200–203}\) especially when perturbative treatment of EPs as well as investigation of high mode numbers made the analysis of modified MHD codes less accurate.\(^{204–209}\) Equation (18) as general framework for computing global TAE dispersion relation and mode structures\(^1\) has been adopted also for realistic ITER equilibria.\(^{210}\) However, with present day computer capabilities, the most efficient way of computing TAE spectra is via direct numerical simulations (cf. Sec. VII). Nonetheless, the asymptotic solution of Eq. (18) as initial value problem\(^{211}\) may allow investigating kinetic physics that are not readily available in existing numerical codes or providing useful code benchmark cases. In fact, it has been shown that the GFLDR formulation of Eqs. (1) and (15) may be very accurate even at low mode number if the actual mode structure is used in the evaluation of potential energies.\(^{189}\) At the same time, benchmarking numerical results against GFLDR predictions with different models of generalized inertia may help assessing the need of kinetic models in numerical simulation codes.\(^{72,73,125,211}\)

We now apply the GFLDR in the form of Eq. (15) to increasingly shorter wavelengths but still neglecting finite thermal ion Larmor radius effects. For \( |k_0 | \rho_L \geq C(\epsilon_0) \), EP dynamics becomes ignorable in the kinetic layer (cf. Sec. IV and Ref. 1) and, thus, \( \beta_{IE} \) can be dropped in the expression of \( \beta_i \), Eq. (64). Wave-EP interactions become gradually more affected and eventually dominated by \( \delta \tilde{W}_i \); first, they are independent of the mode number, while they eventually decrease and vanish due to finite orbit width for \( |k_0 | \rho_L \geq 1 \).\(^{3,4,136,137}\) Thus, most unstable mode numbers are expected for \( C(\epsilon_0) \approx |k_0 | \rho_L \approx 1 \). Similar to Eq. (66), it is possible to demonstrate that the general form of \( \delta \tilde{W}_{nkt} \) is\(^{144}\)

\[
\delta \tilde{W}_{nkt} = 4i \Lambda_{nkt} \left\{ G_f (s, x, \theta_k) \left( \delta \tilde{W}_{nkt}^u + \delta \tilde{W}_{nkt}^\nu \right) + \left[ H_f (s, x, \theta_k) \cos \theta_k + L_f (s, x, \theta_k) \sin \theta_k \right] \delta \tilde{W}_{nkt}^i \right\} + \left[ Z_f (s, x, \theta_k) \delta \tilde{W}_{nkt}^i - \frac{(B(0)^2 - A(0)^2)}{2A(0)^2} \delta \tilde{W}_{nkt}^u \right],
\]

where, for the sake of simplicity, the EP \( \tilde{F}_0 \) has been assumed to be symmetric in \( e_k \). More general expressions of \( \delta \tilde{W}_{nkt} \) are given in Ref. 144. For \( |s|, |x| < 1 \), Eq. (71) becomes

\[
\delta \tilde{W}_{nkt}^i = \delta \tilde{W}_{nkt}^i + \delta \tilde{W}_{nkt}^u \left( \frac{1 - B(0)^2}{A(0)^2} \right),
\]

(72)

Here, \( \delta \tilde{W}_{nkt}^i \) represents TAE mode excitation by trapped particles\(^{4,212}\) and is given by the same expression of \( \delta \tilde{W}_{nkt}^u \) in Eq. (49).\(^4\) Meanwhile, adopting the notation introduced in Ref. 4, \( \delta \tilde{W}_{nkt}^u \) accounts for TAE mode excitation by transit resonances and

\[
\delta T_{nk} = \frac{\pi^2}{8s} \frac{e^2}{\sqrt{m c^2}} \sum_{n} \sum_{\lambda} \left[ \frac{\Omega_{n,\lambda}^2}{\Omega_0^2} \right] Q_{\lambda}^n \Delta_{\lambda}^n (1 + A_{\lambda})^{3/2} \times \frac{e}{c_{\lambda}^2} \left[ \frac{\delta T_{nk}}{\delta t} \right] \right]_{r E}.
\]

(73)

Similar to Eq. (69), Eq. (72) shows that the even parity TAE \((B(0)/A(0) \ll 1)\) is preferentially excited for \( |s|, |x| < 1 \) and it obeys Eq. (15); i.e., the GFLDR\(^4\)

\[
i \Lambda_{nkt} = \frac{1}{2} \frac{B(0)}{A(0)} = \frac{1}{2} \sqrt{\frac{\Gamma_+}{\Gamma_-}} \delta \tilde{W}_{nkt}^i + \delta T_{nk}^u + \frac{|s| \pi}{8} \left( 1 + 2k(s) \cos \theta_k - \frac{x}{\pm k_c} \right),
\]

(74)

From Eq. (15), we can also obtain the dispersion relation for the odd parity TAE \((B(0)/A(0) \gg 1)\), which is typically suppressed and should satisfy

\[
\frac{1}{2} \frac{A(0)}{B(0)} = \frac{1}{2} \sqrt{\frac{\Gamma_-}{\Gamma_+}} = - \frac{\left| s \right| \pi}{8} \left( 1 + 2k(s) \cos \theta_k + \frac{x}{\pm k_c} \right).
\]

(75)

As final application of the GFLDR in the form of Eq. (15), we consider short wavelength modes for which finite thermal ion Larmor radius effects must be taken into account. In fact, for further increasing mode numbers, or when the mode frequency approaches the SAW continuum and TAE mode structures become more singular, \( \propto \rho_L^2 \) terms in Eq. (63) become important. That system of equations, in fact, can be regarded in an approximate sense as the
Schrödinger equation for a particle with energy \( E_{\text{eff}} = \Gamma_{+} \Gamma_{-} \) that moves in a potential well \( V_{\text{eff}} \simeq (\Gamma_{+} + \Gamma_{-}) s^{2} \nabla_{\perp}^{2} p_{K}^{2}/4 - s^{2} \nabla_{\perp}^{4} p_{K}^{4}/16 \). Thus, for modes that are not bounded at sufficiently small \( |\theta_{l}| \) (\( \Gamma_{+} + \Gamma_{-} < 0 \), corresponding to TAE), the \( V_{\text{eff}} \) asymptotic structure is always an anti-well that yields "radiative" damping.\(^{170,171,213-215}\) These modes are kinetic TAE (KTAE) and describe the discretization of the SAW continuum near the TAE gap. Due to the \( s^{2} \nabla_{\perp}^{2} p_{K}^{2}/4 \) contribution in \( V_{\text{eff}} \), lower and upper KTAE branches are not symmetric. While the lower KTAE is strongly damped (\( \Gamma_{+} + \Gamma_{-} < 0 \)), the upper KTAE (\( \Gamma_{+} + \Gamma_{-} > 0 \)) may be significantly bounded by the local well structure at \( s^{2} \nabla_{\perp}^{2} p_{K}^{2}/4 < (\Gamma_{+} + \Gamma_{-}) \) and be affected by radiative damping only via tunneling to higher \( \theta_{l}^{2} \).\(^{170,171,213-215}\) The asymptotic analysis of TAE and KTAE radiative damping is reviewed in Ref. 144. Here, for brevity, we present the extension of the GFLDR to short wavelength KTAE based on the same approach yielding Eq. (56); i.e., on the approximate solution of Eq. (63) near lower and upper SAW continuum accumulation points.\(^{176}\) Considering first the lower accumulation point, \( |\Gamma_{+}| \ll \epsilon_{0} \) and \( \Gamma_{-} \simeq -2 \epsilon_{0} \omega_{0}^{2}/\omega_{A}^{2} \). In this limit, Eq. (63) can be readily solved for \( A(\theta_{l}) \) and \( B(\theta_{l}) \), and yields

\[
\frac{A(0)}{B(0)} = \frac{\exp(-i\pi/4) \Gamma(1/4 + a_{l}/2)}{(2\Delta K)^{1/4} \Gamma(3/4 + a_{l}/2)},
\]

\[
a_{l} = -\frac{\Gamma_{+}}{|s|\rho_{K}/\sqrt{2\epsilon_{0}\omega_{0}^{2}/\omega_{A}^{2}}},
\]

with \( \Delta K = (1/4)s^{2}\rho_{K}^{2}/(\epsilon_{0}\omega_{0}^{2}/\omega_{A}^{2}) \). When substituted into Eqs. (74) and (75), Eq. (76) describe, respectively, even and odd KTAE branches near the lower SAW continuum accumulation point. Given the properties of the Euler \( \Gamma \)-function,

\[
a_{l} = -(2k + 1/2) \quad (\text{even}) \quad \text{and} \quad a_{l} = -(2k + 3/2) \quad (\text{odd})
\]

(77)

describe the lowest-order dispersion relation of the lower KTAE branch with \( k \in \mathbb{N}^{+} \). Expanding \( a_{l} \) defined in Eq. (76) near the lower SAW continuum accumulation point, Eq. (77) can be rewritten as

\[
\frac{\Delta \omega}{\omega_{A}} = -i \sqrt{\frac{2}{\epsilon_{0}}} |s|\rho_{K}(2k + \delta),
\]

(78)

where \( \Delta \omega \) is the lower KTAE frequency shift; and \( \delta = 1/2, 3/2 \) for even/odd modes, respectively. Thus, Eq. (78) provides the expression for the strong radiative damping anticipated above for this branch. Similarly, near the upper SAW continuum accumulation point, it is possible to show that

\[
\frac{A(0)}{B(0)} = (2\Delta K)^{1/4} \Gamma(3/4 + a_{l}/2) / \Gamma(1/4 + a_{l}/2),
\]

\[
a_{l} = -\frac{|s|\rho_{K}/\sqrt{2\epsilon_{0}\omega_{0}^{2}/\omega_{A}^{2}}};
\]

(79)

which admit the lowest-order solutions

\[
a_{u} = -(2k + 3/2)(\text{even}); \quad \text{and} \quad a_{u} = -(2k + 1/2)(\text{odd}).
\]

(80)

Again, expanding \( a_{u} \) defined in Eq. (79) near the upper SAW continuum accumulation point, Eq. (80) can be rewritten as

\[
\frac{\Delta \omega}{\omega_{A}} = \sqrt{\frac{2}{\epsilon_{0}}} |s|\rho_{K}(2k + \delta),
\]

(81)

with \( \delta = 1/2, 3/2 \) for odd/even modes, respectively. Thus, Eq. (81) confirms that the upper KTAE branch is affected by radiative damping only via tunneling to short wavelengths;\(^{170,171,213-215}\) and that finite thermal ion Larmor radii, at the lowest order, merely provide a real frequency shift.

In analogy with Eq. (56) and following discussions, Eqs. (76) and (79) show that KTAE as discrete structures ("granularity") of the SAW continuum depend on the spatiotemporal scales on which they are "observed."\(^{144,176}\) This is shown in Fig. 1, reporting numerical solutions of two

FIG. 1. Evidence of merging of one KTAE into an EPM above the EPM destabilization threshold when the spatiotemporal properties of the mode make the discretized SAW spectrum behave as a "true" continuum (from the original Figs. 3(a) and 3(b) in Ref. 144). Reprinted with permission from Phys. Plasmas 3, 323 (1996). Copyright 1996, AIP Publishing LLC.
neighbor roots of Eqs. (74) and (75) with \(A(0)/B(0)\) given by Eq. (79); i.e., two neighboring KTAE modes (crosses and open squares). These roots are compared with the only one (open circles) obtained from the same equations with \(\Delta E = 0\), representing an EPM \(\text{EPM}^1\) (cf. Sec. VI). For all three modes, growth rates, represented by \(\text{Im}(\Gamma_\omega \omega_c^2/\epsilon_0 \omega_c^2)\) in the left frame, and real frequencies, given by \(\text{Re}(\Gamma_\omega \omega_c^2/\epsilon_0 \omega_c^2)\) in the right frame, are shown vs. the energetic particle \(\Delta E\), defined as in Eq. (37) with \(\beta \rightarrow \beta_E\). It is evident that normalized KTAE frequency and growth rates are weakly modified \((-O(10^{-1})\)) by EPs below the EPM destabilization threshold at \(\Delta E \approx 0.18\). Above that threshold, one of the two KTAE (crosses) behaves as and becomes the EPM. This shows that, when one of the KTAE responds more strongly (normalized mode frequency and/or growth rate change \(\geq 0.5\)) and its growth rate becomes larger than the characteristic frequency separation of the KTAE spectrum [in normalized units \(-O(10^{-1})\)], it “feels” the presence of the other KTAEs (the discretized SAW continuum) as a “true” continuous spectrum (cf. Sec. IV).

VI. ENERGETIC PARTICLE MODES

In Sec. III, we have analyzed both branches of the fishbone oscillations predicted by the GFLDR; \(\text{EPM}\) i.e., the EPM branch, investigated for the first time in Ref. 2 and generally dubbed as precessional fishbone; and the AE or gap mode branch, addressed in Ref. 34 and known as diamagnetic fishbone. Sections IV and V, meanwhile, have addressed various types of AE fluctuation branches, including KBB, BAAE, BAE, RSAE, TAE, and their kinetic variants. In this section, we complete the examples of GFLDR solutions discussing the EPM branch near the TAE frequency gap, which was already introduced in Fig. 1 as illustration of the transition from KTAE to EPM modes for increasing strength of the EP drive above the EPM excitation threshold. We note that the EPM mode can be excited in the whole frequency range between the precessional fishbone and the TAE gap. In fact, EPMs are excited when the EP drive is sufficiently strong to overcome continuum damping, as shown by the GFLDR, Eq. (1), yielding the threshold condition, Eq. (31). For radially localized EPs, meanwhile, the EP dispersion relation is given by the GFLDR in the form of Eq. (15); and the threshold condition, Eq. (31), is modified accordingly. For frequencies in the range between the precessional fishbone and the TAE gap, the various expressions of \(\Delta \omega\), \(\delta W_n\), and \(\delta W_m\) derived in Sec. IV are readily applicable.\(^4\)

As emphasized in the companion paper, Ref. 1, the relevance of EPMs stems from their nature of being born as unstable discrete modes out of the SAW continuous spectrum at the optimal frequency for maximizing wave-EP power exchange. For this reason, above the linear EPM excitation threshold, there is a transition from local to meso-scale particle redistributions, as noted in Ref. 216 and discussed in Ref. 10. The non-perturbative nature of EPMs is reflected by the sensitivity of mode frequency and growth rates to EP sources. Thus, linear EPM dispersive properties suggest that fast frequency chirping is to be expected for these modes during their nonlinear evolution, as noted in Refs. 10, 208, 217, and 218.

The excitation of EPMs is generally independent of the existence of frequency gaps in the SAW continuum, although Eq. (31) shows that EPM threshold is lower near an accumulation point. For this reason, sometimes EPMs have been given specific names; such as Resonant TAE (RTAE) modes\(^\text{200}\) for EPM near the TAE frequency gap, as those considered in this section. In some special conditions, EPM has also been used as acronym for indicating fluctuations observed experimentally and requiring non-perturbative EP responses for AE to exist inside frequency gaps; e.g., RSAE/\(\text{AE}\)\(^\text{151,152}\) excited by highly supra-thermal ion tails due to ion cyclotron resonance heating (ICRH). Given the GFLDR theoretical framework, this use is not justified and is reported here for the sake of completeness.

After the first theoretical work on EPMs,\(^4\) numerical analyses of EPM stability were not numerous \(\text{144,200,203,208,210,218,219}\) until the first hybrid MHD-gyrokinetic simulation demonstrated the potential severe impact of these modes on EP confinement,\(^\text{216}\) consistent with experimental observations. Ever since, significant efforts have been devoted in comparing numerical simulation results with experimental observations of EPMs\(^\text{220–222}\) (cf. Sec. VII for more details).

A case that is particularly simple and still retains all the necessary physics ingredients to elucidate linear dispersive properties and radial structures of EPM is that of precessional resonance with magnetically trapped EPs without finite orbit width effects. Using Eq. (25) for separating the fast particle convective response, which is included via \(\delta W_n\) in the GFLDR dispersion relation, Eq. (15), Eq. (27) is readily solved for the EP bounce averaged response. In the case of plasma equilibria with shifted circular magnetic surfaces and assuming that particles are deeply trapped\(^3,4\) (cf. Sec. IV), \(\mathcal{J}B_0 \approx qR_0, n\omega_d = -k_0 \varepsilon/(R_0 \Omega_E), \tau_B = 2\pi qR_0/(R_0)^{1/2}\varepsilon^{-1/2}\), and, consistent with Eq. (49),

\[
\delta W_n = \left| E \delta \mathcal{E} \delta \mathcal{E} \sum_{\varepsilon_i} \frac{\pi^2 \varepsilon_i R_0}{2 \varepsilon_i^2 E_0} \sum_{m} \left( \frac{\tau_B^2 \omega_c^2 Q_{E,N} R_0}{n\omega_d - \varepsilon_i} \right) \right. 
\]

(82)

Consider now one single EP species with an isotropic slowing down distribution function, characterized by injection energy \(E_F\) much larger than the critical energy \(E_c\),\(^\text{223}\) so that for \(E_F/m\varepsilon > E_c/m\varepsilon\) the EP energy is predominantly transferred to thermal electrons by collisional friction as it occurs for \(\varepsilon\)-particles in fusion plasmas; i.e.,

\[
\mathcal{F}_0 = \frac{3P_{\varepsilon F}}{4\pi E_F} \left( \frac{H(E_F/m\varepsilon - E)}{2E_c^{3/2} + (2E_c/m\varepsilon)^{1/2}} \right),
\]

(83)

where \(H\) denotes the Heaviside step function and the normalization condition is chosen such that the EP energy density is \((3/2)P_{\varepsilon F}\) for \(E_F \gg E_c\). Substituting Eq. (83) back into Eq. (82), we obtain\(^\text{224}\)

\[
\delta W_n = \frac{3\pi (r/R_0)^{1/2} \Delta E}{8 \sqrt{2} |s|} \left[ 1 + \frac{\omega}{\omega_n} \ln \left( \frac{\omega_n}{\omega} - 1 \right) + i \pi \frac{\omega}{\omega_n} \right],
\]

(84)

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with $\omega_{df} = \omega_c (E = E_F/m_E)$ and having noted that $|\omega_E| > |\omega|$ for SAW resonantly excited by EPs. Thus, EP induced wave-particle power exchange is mainly via gradients in the radial profiles of the EP component. This result is readily applied to investigate the linear EPM excitation by deeply trapped EPs near the TAE gap, where, for moderate values of $(s, x)$, Eq. (68) yields
\[
\delta \tilde{W}_TF = \left[ \frac{|s|}{8} \left( 1 + 2 \kappa(s) - \frac{x}{\Delta_T} \right) + \frac{|s|}{8} \kappa(s) \frac{1}{k^2_{\Delta E}} \frac{\partial^2}{\partial_s^2} \frac{1}{\Delta_T^2} \right], \quad (85)
\]
where we have assumed $|B(0)/A(0)| \ll 1$ and $\cos \theta_b \approx 1 - \theta_b^2/2 = 1 + (1/2)(s_k)^2 \frac{\kappa^2}{\kappa^2 + 2}$. The EP radial profile effect on EPM mode structures can be analyzed by assuming that their equilibrium pressure gradient is peaked at $r = r_0$; i.e.,
\[
\frac{\kappa}{\Delta_T} = \frac{\kappa}{\Delta_T} \left[ \frac{1}{1 + \Delta_T^2} \right] \approx \frac{\kappa}{\Delta_T} \left[ \frac{1}{1 + \Delta_T^2} \right] \frac{1}{1 - \Delta_T^2}, \quad (86)
\]
with $\lambda = |s| \kappa_0 (r - r_0)$ and other equilibrium quantities are considered constant for localized modes. Thus, dropping the subscript $n$ and letting $\omega_{df} \equiv n \omega_{df}$, Eq. (74) finally becomes
\[
i \Delta TA = \left[ \frac{|s|}{8} \left( 1 + 2 \kappa(s) - \frac{x}{\Delta_T} \right) + \frac{|s|}{8} \kappa(s) \frac{1}{k^2_{\Delta E}} \frac{\partial^2}{\partial_s^2} \frac{1}{\Delta_T^2} \right] \frac{\pi}{8 \sqrt{2}|s|} \left[ \frac{3 \pi (r/R_0)^{1/2}}{8 \sqrt{2}|s|} \kappa(s) \frac{1}{k^2_{\Delta E}} \frac{\partial^2}{\partial_s^2} \frac{1}{\Delta_T} \right] \frac{\pi}{8 \sqrt{2}|s|} \left[ \frac{3 \pi (r/R_0)^{1/2}}{8 \sqrt{2}|s|} \kappa(s) \frac{1}{k^2_{\Delta E}} \frac{\partial^2}{\partial_s^2} \frac{1}{\Delta_T} \right] = 0,
\]
\[
\frac{\gamma_T}{\omega_0} = \left[ \frac{\omega_0}{\omega_{df}} - \frac{\omega_0}{\omega_{df}} \ln \left( \frac{\omega_{df}}{\omega_0} - 1 \right) \right] \pi \frac{\omega_0}{\omega_{df}},
\]
\[
\gamma_T = \left[ \frac{\omega_0}{\omega_{df}} - \frac{\omega_0}{\omega_{df}} \ln \left( \frac{\omega_{df}}{\omega_0} - 1 \right) \right] \pi \frac{\omega_0}{\omega_{df}} \left[ \frac{3 \pi (r/R_0)^{1/2}}{8 \sqrt{2}|s|} \kappa(s) \frac{1}{k^2_{\Delta E}} \frac{\partial^2}{\partial_s^2} \frac{1}{\Delta_T} \right] \frac{\pi}{8 \sqrt{2}|s|} \left[ \frac{3 \pi (r/R_0)^{1/2}}{8 \sqrt{2}|s|} \kappa(s) \frac{1}{k^2_{\Delta E}} \frac{\partial^2}{\partial_s^2} \frac{1}{\Delta_T} \right] = 0.
\]

Equations (88) and (89) summarize all the characteristic features of EPs, namely, their radial localization is determined by the EP source via wave-particle resonant drive. Equation (88) also shows that the most unstable EPM wave packet is excited on meso-scales $\sim (L_{\Delta E}/|k_0|)^{1/2}$. Furthermore, the real EPM frequency is controlled by the EP characteristic frequency, in this case $\omega_{df}$, while the instability threshold condition is clearly set by the competition between EP drive and continuum damping. The crucial role of these EPM properties in the nonlinear evolutions of these fluctuations is discussed in Refs. 10 and 217.

VII. EXPERIMENTAL OBSERVATIONS AND THEORETICAL UNDERSTANDINGS

The many experimental observations of AE and EPM are well documented and discussing them in detail is beyond the scope of the present work. Here, we illustrate only some of the successful and positive feedbacks between theory and experiment in this area; made possible by the development of impressive diagnostic techniques as well as numerical simulation capabilities, accompanied by detailed physics understanding that are readily explained within the GFLDR theoretical framework.
diamagnetic direction in the presence of an inverted spatial gradient of the supra-thermal tail, consistent with Eq. (23). E-fishbone properties have been characterized by very detailed analyses of experimental results in HL-2A, FTU, and Tore Supra, based on the use of the GFLDR as an interpretative framework, and confirmed by hybrid MHD-gyrokinetic numerical simulations.

The first observations of AE gap modes were in the TAE frequency range, reported independently in TFR and DIII-D. The different varieties of AE associated with various equilibrium geometry and nonuniformity became evident already with D-T experimental results in TFTR observing a core localized TAE, readily explained with by theoretical studies. The high performance D-T experiments in JET were instead stable with respect to those expected in burning plasmas. In particular, the relatively large fusion alpha particle orbit widths in today's machines causes their response to be nearly adiabatic, and, thereby, suppress resonant wave-particle interactions.

One evident consequence of the GFLDR is the general feature of AE in toroidal plasmas to be slightly shifted from the SAW accumulation point; i.e., to closely track the frequency of the continuous spectrum at the SAW resonance. This makes it possible to infer local plasma parameters by “MHD spectroscopy,” which measures the AE frequency. MHD spectroscopy has been particularly adopted as analysis technique in connection with experimental observation of AC/RSAE fluctuations. In addition to the generic information on the Alfvén speed and the plasma rotation by Doppler shift, the observation of TAE and AC/RSAE may yield information on the q profile, which is especially useful when the time evolution of the minimum-q can be reconstructed, for it helped developing plasma operation scenarios with internal transport barriers. MHD spectroscopy has been proposed as (thermal ion) temperature diagnostics as well, using the temperature ratio dependence of the BAE accumulation point (cf. Sec. IV).

One important progress in comparisons between experimental observations and theoretical prediction has been driven in the recent years by the development of internal measurements of AE mode structures. DIII-D is the first to demonstrate that AE may be excited in the plasma core by both EPs as well as thermal ions with a wide range of mode numbers, consistent with the theoretical framework of Sec. IV. Other examples are the use of reflectometry and Phase Contrast Imaging (PCI) techniques, providing good comparisons of 2D MHD and gyrokinetic mode structure calculations with experimental measurements, and of Electron Cyclotron Emission (ECE) imaging, yielding a visualization of actual 2D AE structures from experimental measurements. ECE imaging results of RSAE mode structures are found to be influenced by the the EP radial profile, as expected from theory and in agreement with gyrofluid (cf. Fig. 2) and gyrokinetic simulation results. Qualitatively similar results of 2D RSAE mode structures, modified by the EP radial profiles, have been obtained by gyrokinetic (cf. Fig. 3) and hybrid MHD gyrokinetic simulations. Note that the GFLDR theoretical framework contains effects of thermal plasma and EP radial profiles as well as wave-particle resonances; thus, the corresponding up-down symmetry in the parallel mode structures will, in general, be broken. Such symmetry-breaking conditions are favored by localized EP drive, moderate mode numbers and/or by vanishing magnetic shear, since the characteristic mode width due to q profile then scales as \( q_n \). Further discussions of these issues and of the validity of Eqs. (4) and (18) for moderate mode numbers can be found in Ref. 23. Nonperturbative EP effects in determining also TAE mode structures have been recently reported in gyrokinetic simulations, consistent with the GFLDR theoretical framework and the expected smooth transition between AE and EPM branches of SAW fluctuations driven by EPs.
In general, however, good agreement between internal measurements of AE mode structures and numerical calculations are not always found. This is the case, for example, of “TAE Avalanches,” where the discrepancies between mode structures reconstructed from reflectometry and numerical simulation results are attributed to nonlinear processes. Furthermore, fluctuations are typically accompanied by rapid frequency chirping. These observations, indeed, suggest that supra-thermal particle transport and nonlinear SAW/DAW dynamics are profoundly interlinked, and that frequency chirping is an important nonlinear process, whose characteristic rate provides information on the underlying physics.

The low-frequency SAW wave spectrum in the KTI frequency gap also attracted significant interest since the first observations of BAE modes. A very detailed stability analysis of BAE modes observed in Tore Supra and of the experimental conditions that are necessary for effective mode excitation is given by Nguyen et al. on the basis of the GFLDR theoretical framework (cf. Sec. IV). The GFLDR was also adopted for explaining the frequency of BAE modes in FTU, where they are excited in the presence of a large magnetic island, as also observed in TEXTOR and HL-2A. For accurate interpretation of these observations, as shown in Sec. IV, it is necessary to use kinetic theories for the proper treatment of thermal plasma compression effects, including wave-particle interactions with circulating as well as trapped thermal plasma particles. Recent detailed gyrokinetic simulations and comparisons with experimental observations in ASDEX Upgrade have confirmed that accurate description of kinetic interactions are needed for capturing the physics of AE and EPM at frequencies near the KTI gap. These results also confirm that two bands of low-frequency SAW/DAW activities are generally expected, with predominance of either ion diamagnetic drift (KBM) or parallel and perpendicular ion compressibility (BAE) and with varying frequency-dependent geodesic curvature coupling to the ion-acoustic wave. More recently, the observation of BAE modes driven by supra-thermal electrons (e-BAE) due to ECRH near the KTI frequency gap in HL-2A has renewed the interest on the particular role that investigating fast-electron driven DAWs may have in understanding burning plasma physics. Observations of e-BAE in HL-2A can also be understood within the GFLDR theoretical framework.

The rapid increase of numerical simulation capabilities and the emergence of a unified theoretical framework for understanding and analyzing the excitation of DAWs by EPs suggest that realistic burning plasma stability analyses will be possible in the near future. Furthermore, all numerical simulation activities of Alfvénic modes in ITER are not only aiming toward stability analyses. In fact, significant efforts are also going on, e.g., for providing technical support to the development of optimized diagnostic systems, such as those discussed in Ref. 272.

VIII. CONCLUSIONS AND DISCUSSIONS

In this work, on the basis of the companion paper, Ref. 1, we have demonstrated that the GFLDR theoretical framework provides a unified description for SAWs/DAWs excited by EPs in toroidal fusion plasmas. The GFLDR can be used for analytical as well as numerical calculations based on different models and with different levels of approximation, as it is shown by detailed applications discussed here for cases of practical interest. The GFLDR by itself does not provide additional information with respect to that included in the gyrokinetic vorticity equation and quasineutrality condition, on which it is based and that are rather general.
However, it suggests a unified framework for extracting the underlying physics of observed fluctuations and for their classification as either AEs or EPMs. Meanwhile, the GFLDR allows to identify characteristic spatiotemporal scales of processes involved; and, in general, to adopt different approaches to their investigation where appropriate. In this way, the GFLDR elevates the interpretative capability for both experiments and numerical simulations; and clarifies the importance of kinetic descriptions and of accurate geometries and boundary conditions, which must be taken into account for predicting linear as well as nonlinear SAW/DAW and EP behaviors in burning plasmas.

Applications and examples presented in this work deal with linear physics, while in depth discussions of nonlinear dynamics of SAW/DAW excited by EPs are given in Ref. 10. In particular, we have investigated mode dispersion relations, stability properties and discussed mode structures of SAW/DAW fluctuations in the frequency range from fishbones to TAEs. Once again, the same results may be well obtained by direct numerical solutions of gyrokinetic vorticity equation and quasineutrality condition. The advantage of analyses based on the GFLDR stands in the physical interpretation of underlying processes within a unified framework. Thus, the GFLDR provides not only a practical approach but a conceptual methodology as well.

As final remark, we note that the GFLDR is based not only on the underlying nonlinear gyrokinetic theoretical background but also on the existence of two characteristic radial scales of DAW fluctuations; which is generally the case of low frequency MHD and SAWs excited by EPs in nonuniform toroidal plasmas. Thus, the GFLDR is generally not applicable in cases where this separation of scales is not present. For example, it cannot be adopted for investigating strong DAW turbulence with \(|\gamma_{\parallel}/c_0| \sim 1|\); and it is also not applicable to AEs, dubbed \(z_{\text{TAE}}\),\(^{273}\) existing in the high-\(\beta\) ideal ballooning mode “second stability region,”\(^{274}\) where \(z \gg 1\) and \(z\) is the normalized pressure gradient of Eq. (37). More specifically, \(z_{\text{TAE}}\) have regular mode structures with exponentially small coupling to the continuum, and, hence, GFLDR cannot be applied. These modes have clear connection with the higher order collisionless ballooning modes discussed in Ref. 275, whose stability boundaries were predicted by Chen et al.,\(^{276}\) and investigated in Ref. 277. Similar to other AEs, \(z_{\text{TAE}}\) can also be destabilized by an EP population.\(^{278}\) The dispersion relation of these modes and their mode structures have been studied in detail in recent numerical simulation works,\(^{279,280}\) where a quadratic form similar to Eq. (1) is derived and \(\Lambda_0\) can be interpreted as the rate of energy leaking to smaller scales. There, it is demonstrated that, including thermal plasma kinetic effects, \(z_{\text{TAE}}\) can be viewed as modified Alfvénic ITG in the second stability region; i.e., the modes first addressed in Ref. 275 and then in Ref. 277. Meanwhile, in the presence of EP drive, \(z_{\text{TAE}}\) acquire their true nature of AEs destabilized by a sparse supra-thermal particle population. The relevance of \(z_{\text{TAE}}\) or other type of drift-Alfvén ballooning modes, exhibiting similar periodic stability patterns,\(^{276}\) to present day toroidal devices is limited, for they require reaching \(z \approx 1\). It may, however, be relevant to the concept for DEMO (DEMOnstration Power Plant); should high-\(\beta\) plasma operations be the preferred option for DEMO among those presently under discussion.

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