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Kinetic Theories of Geodesic Acoustic Modes: Radial Structure, Linear Excitation by Energetic Particles and Nonlinear Saturation∗

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Abstract Geodesic acoustic modes (GAMs) are oscillating zonal mode structures unique to toroidal plasmas and are capable of regulating microscopic turbulence and associated transports. In this paper, three important aspects of GAM dynamics are investigated, namely (1) GAM continuous spectrum and its mode conversion to kinetic GAM (KGAM); (2) linear excitation of energetic particle induced GAM (EGAM) and its coupling to the GAM continuum, and (3) nonlinear saturation of EGAM via wave particle trapping. The analogy between the GAM-EGAM dynamics and the well known beam-plasma instability are also discussed.

Keywords: geodesic acoustic mode, continuum, energetic particles, wave particle trapping

PACS: 52.35.Fp, 52.35.Qz, 52.55.Tn

1 Introduction

Geodesic acoustic modes (GAMs) are toroidally symmetric normal modes unique to toroidal plasmas with a mode structure that is nearly poloidally symmetric. GAMs are predominantly electrostatic modes, characterized by an n = 0/m = 0 scalar potential, and an n = 0/m = 1 up-down antisymmetric density perturbation. Here, n/m are the toroidal/poloidal mode numbers of the torus. They occur since the charge separation effect, due to the ion radial magnetic drift associated with geodesic curvature, causes a finite parallel a.c. electric field (n ξ/ξT) and perturbed ion diamagnetic current and polarization current to ensure quasi-neutrality via electron and ion dynamic responses, respectively. GAMs have received much attention in the magnetic fusion plasma research due to their potentially important roles in regulating drift waves, and, hence, transports via nonlinear interactions.

Energetic particles (EPs), generated as charged fusion products or by auxiliary heating, such as neutral beam injection and/or ion cyclotron resonance heating (ICRH), are of much more higher energy than the bulk thermal ions of the plasma. As plasma oscillations interact resonantly with the characteristic periodic motions of the EPs (for example, the EPs’ transit/bounce/precession frequency), they can be driven unstable when the drive provided by EPs exceeds the damping of the fluctuations themselves. GAMs driven by EPs have been observed in recent experiments and the stability properties of this energetic-particle-induced GAM (EGAM) were analyzed theoretically in Ref. [4].

This paper is organized as follows. In section 2 the GAM mode equation via fluid theory is derived, the existence of the GAM continuous spectrum is demonstrated and the physics associated with it, such as phase mixing and mode conversion to kinetic GAM (KGAM) are studied in detail. In section 3 the dispersion relation of EGAM is derived, the dependence of the EGAM radial mode structure on the radial localization of EPs and its coupling to the GAM continuum are studied. In section 4, wave particle trapping is studied to explain the nonlinear saturation of EGAM and how this may yield higher order harmonic generation. Section 5 is devoted to qualitative discussions of analogies and differences of GAM/EGAM physics with the well-known case of a supra-thermal electron beam propagating in a strongly magnetized one-dimensional (1D) plasma. Finally, conclusions and discussions are given in section 6.

2 GAM continuous spectrum and mode conversion to KGAM

We will first derive the mode equation of GAM via fluid theory to demonstrate the GAM continuous spectrum, and study the rich phenomenons associated with the GAM continuous spectrum. Here, a large aspect-ratio axisymmetric tokamak with straight field line flux coordinates (r, θ, ξ) is considered, with the equilibrium magnetic field given by Bθ = Bθ0[ξ/(1 + ϵ cos θ)] + (ϵ/q)ξ, where ξ, q and θ are, respectively, toroidal and poloidal angle-like flux coordinates of the torus and ϵ = r/R0 is the inverse aspect

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ratio. We limit our discussion in this paper to the electrostatic case, since the electromagnetic perturbation, with dominant $m = 2$ poloidal mode structure, will not contribute to the GAM dispersion relation. Readers interested in the magnetic perturbations of GAM may refer to Refs. [2,13].

We start with the linearized fluid equations,

$$m_n v_0 \frac{\partial \delta v}{\partial t} = -\nabla \delta P + \frac{\delta J \times B_0}{c},$$  \hspace{1cm} (1)

$$\frac{\partial}{\partial t} \rho \delta v + \nabla \cdot (n_0 \delta v) = 0,$$  \hspace{1cm} (2)

$$\delta P = \gamma_n T_e \delta n_e + \gamma_i T_i \delta n_i,$$  \hspace{1cm} (3)

$$\delta J + \delta v \times B_0/c = 0,$$  \hspace{1cm} (4)

where Eqs. (1)–(4) are linearized momentum equation, continuity equation, equation of state and Ohm's law respectively, while $m, n, v, P, J, \gamma, T, B$ and $E$ denote the mass, density, velocity, pressure, current, adiabatic index, temperature, magnetic field and electric field respectively, subscripts $e, i$ denote, respectively, electron and ion species.

GAM is described by the flux-surface averaged quasineutrality condition, which reads (locally)

$$\frac{\partial}{\partial R} \langle \delta J_r \rangle = 0,$$  \hspace{1cm} (5)

where $\langle \cdots \rangle$ denotes flux-surface averaging and $\delta J_r$ is the fluctuating radial current, which can be obtained from Eq. (1) as:

$$\delta J_r = \frac{c}{B} \left[ n_0 m_i \frac{\partial \delta v}{\partial v} + \frac{1}{r} \frac{\partial}{\partial \theta} \delta P \right];$$  \hspace{1cm} (6)

and is made up of the polarization current and the perturbed diamagnetic current due to the perturbed pressure gradient in poloidal direction. The $\delta n_\theta = c \delta E_r / B_0$ term is the electric drift induced by the radial electric field, the compressibility of which, due to the toroidal effects causes an $m = 1$ poloidal density perturbation:

$$\delta n = -\frac{c n_0 k_r \delta \phi \sin \theta}{\omega B_0 R_0}.$$  \hspace{1cm} (7)

Combining Eqs. (3), (5), (6) and (7), the radial GAM mode equation can be obtained as:

$$\frac{\partial}{\partial R} \left[ \frac{c^2}{B_0^2} m_n \omega \left(1 - \frac{\gamma_n T_i + \gamma_i T_e}{\omega^2 R_0^2 m_i} \right) \right] \frac{\partial}{\partial R} \delta \phi = 0.$$  \hspace{1cm} (8)

Eq. (8) is identical to that describing shear Alfven wave (SAW) resonance [9] and thus it demonstrates that GAM also constitutes a continuous spectrum described by $\omega = \omega_G(r) [6]$, with $\omega_G \equiv \sqrt{(\gamma_n T_i + \gamma_i T_e)/(m_i R_0^2)}$ being the lowest order GAM frequency in the fluid description. The key feature of the continuous spectrum is that the fluctuations consist of radial singular structures. Meanwhile, rich physics are associated with the plasma nonuniformity, including the phase mixing of initial perturbations [8], resonant absorption of driven instabilities [9] and resonant mode conversion to short wavelength kinetic modes [11].

If one integrates Eq. (8) in the radial direction, then take the time derivative and note $\delta E_r = -\partial_r \delta \phi$, we get

$$\left( \frac{\partial^2}{\partial r^2} + \omega_G^2(r) \right) \delta E_r = S(r, t),$$  \hspace{1cm} (9)

where the right hand side of the equation, the integration constant $S(r, t)$, can be itself a functional of $\delta E_r$, reflecting "slow" radial dependences, and accounts for non-uniform sources and boundary conditions, e.g., the coupling to an external antenna or drive by drift wave turbulence or EPs. As is pointed out in Ref. [6], the global mode structure and frequency of GAM can only be determined when there exists such an external source [11] and when the GAM continuum is properly accounted for. Solving an eigenmode problem in absence of a source/driving term cannot properly analyze the GAM/KGAM mode structure, as indicated in Ref. [6]. This point will be further addressed in section 3.1, where the case with a source term due to velocity space anisotropy of the EP distribution function is analyzed in detail. In the rest of this section, we will discuss the rich phenomena associated with the GAM continuum spectrum and demonstrate the exact analogy of the GAM continuum to the SAW continuous spectrum.

### 2.1 Time evolution of an initial perturbation

We first investigate, in the absence of an external source, the temporal evolution of an initial perturbation. Assuming an initial perturbation $\delta E_G(r)$, with a characteristic scale length much longer than the ion Larmor radius, we can solve Eq. (9) and get the perturbed electric field as

$$\delta E_r = A_1 \exp(-i \omega_G(r)t) + A_2 \exp(i \omega_G(r)t),$$  \hspace{1cm} (10)

in which $A_1$ and $A_2$ can be determined from the initial conditions. The mode will oscillate at the local GAM continuum frequency. Due to the frequency difference at different radial positions, the initial perturbation will generate finer radial structures, as shown in Fig. 1, with corresponding radial wavenumber increasing with time as

$$|k_r| \equiv | \partial_r \delta E_r / \delta E_G | \sim |(d \omega_G / dr)| t.$$  \hspace{1cm} (11)

Thus, $|k_r| \rightarrow \infty$ at every point inside the continuous spectrum as $t \rightarrow \infty$, so the mode structure becomes singular at every point. Eq. (11) can be viewed as a physical manifestation of phase mixing [12], as also noted in numerical simulations [14]. The amplitude of the perturbation, $\delta \phi$ will decay asymptotically in time as $\propto (1/t) \exp(-i \omega_G(r)t)$ [5,12],

$$|\delta \phi| = \left| \int \delta E_r \, dr \right| \propto |1/t|,$$  \hspace{1cm} (12)

and the dependence of $|\delta \phi|$ on time is demonstrated in Fig. 2.
becomes the singularity (FLR) effect, must be taken into account and regularize and the kinetic effects, such as the finite Larmor radius. The dashed line is the fitting with $1/\omega$. The solid line is the amplitude of $\deltaE_r(r,t)$ at four different moments.

![Radial mode structure of $\deltaE_r(r,t)$ at four different moments](image)

Fig. 1

As $|k_r| \to \infty$, the fluid approximation breaks down and the kinetic effects, such as the finite Larmor radius (FLR) effect, must be taken into account and regularize the singularity [6,10,11]. The GAM mode equation becomes

$$\frac{\partial}{\partial r} \left[ \frac{c^2}{B_0} m_i n_0 \omega \left(1 - \frac{\omega^2_G(r)}{\omega^2} (1 + \alpha k_r^2 \rho_L^2) \right) \right] \frac{\partial}{\partial \phi} = 0,$$  

(13)

with $\rho_L$ the ion Larmor radius. Eq. (13) describes the mode conversion of GAM to short wave-length kinetic GAM (KGAM) at $\omega_G(r_c) = \omega$, in analogy with SAW mode conversion to its kinetic counterpart (kinetic Alfvén wave, KAW) [11]. The value of $\alpha$ is positive for typical plasma parameters, so the resultant KGAM will propagate at lower frequencies with respect to the local GAM continuum frequency, i.e., toward the lower temperature and/or higher $q$ region, which is usually outward (Of course, if we consider a reversed shear plasma, when the negative shear is sufficiently large, the GAM continuum will not be monotonically decreasing with the minor radius; instead it can be hollow as a consequence of the hollow $q$ profile). The dependence of the sign of $\alpha$ on the plasma parameters, such as $T_e/T_i$ and $q$, are discussed in Ref. [6].

From Eq. (13), the equation describing the temporal evolution of GAM’s initial perturbation, becomes a wave equation

$$\left[ \alpha \omega^2_G \rho_L^2 \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \phi^2} - \omega^2_G(r) \right] \deltaE_r = 0. \quad (14)$$

For a given initial perturbation with a scale length much longer than the ion Larmor radius, the first term in Eq. (14) is $O(k_r^2 \rho_L^2)$ smaller and, hence, is not important compared to other terms. The initial perturbation, however, will generate short wavelength structures as was discussed following Eq. (9). When the radial scale length is comparable with the ion Larmor radius (with $k_r \rho_L \sim 1$), the first term becomes comparable with the others, and Eq. (14) describes the generated short wavelength structures propagating radially outward with group velocity $v_g = \delta k_r / \delta \omega \propto k_r$. Thus, Eq. (14) describes a long wavelength GAM mode transforming/ converging to a short wavelength KGAM mode [12].

### 2.2 Continuum damping of GAM and mode conversion to KGAM

If one considers a system driven by an external source, with a characteristic frequency $\omega_0$, i.e., $S(t) = S_0 \exp(-i \omega_0 t)$, finite amplitude waves can then be excited. We first ignore the FLR effect, and the mode equation, Eq. (9), can be solved as a boundary value problem yielding

$$\deltaE_r = \frac{S_0 \exp(-i \omega_0 t)}{\omega^2 - \omega^2_G(r)}. \quad (15)$$

Here, $\deltaE_r$ goes to infinity at the radial position $r_c$, where the local GAM continuum frequency equals the driving frequency, corresponding to the GAM continuum resonantly driven by an external drive. The mode amplitude is proportional to $1/(r - r_c)$ near the resonant point $r_c$. The mode is singular at $r_c$, which causes finite absorption of the driven mode by the plasma [8,15]. The power locally absorbed by the plasma is

$$P_{\text{heating}} \propto \text{Re}[\deltaE_r^* \deltaI]$$

$$= - \frac{n_e c}{\Omega_i B_0 \omega} \text{Im}[\deltaE_r^* (\omega^2_G - \omega^2) \deltaE_r]$$

$$= - \frac{n_e c}{\Omega_i B_0 \omega} |S_0|^2 \text{Im}\left[ \frac{1}{(\omega^2_G(r) - \omega^2)^*} \right]. \quad (16)$$

The absorption rate is proportional to the imaginary part of $1/(\omega^2_G(r) - \omega^2)^*$, with the superscript “+” indicating the complex conjugate. The imaginary part can be determined from the Plemij’s formula, i.e.,

$$\frac{1}{(\omega^2_G - \omega^2)^*} = P \left[ \frac{1}{(\omega^2_G - \omega^2)^*} - it \pi \frac{\delta(r - r_c)}{\partial \omega^2_G(r) / \partial r |_{r_c}} \right]. \quad (17)$$
in which, $P$ denotes the principal part in the integration. From the absorption rate

$$P_{\text{heating}} \propto \frac{n_0 e c |S_0|^2 r}{\Omega B_0 \omega \partial \omega^2_\delta(r)/\partial r |_{r_\gamma}} \delta(r-r_\gamma),$$ (18)

it is seen that absorption takes place at the resonant point, with the absorption rate inversely proportional to the scale length of the GAM continuum, in a strict analogy with the SAW case. This physical picture of resonant absorption is essentially the same as that for the (inverse) Landau damping. The only difference is that the resonant absorption happens in the real space where the local continuum frequency resonates with the external driver; while Landau damping happens in the velocity space, where the mode resonates with the continuum formed by particles with different parallel speeds with respect to the wave phase velocity.

At the resonant point, where the radial mode structure is singular and the wavelength is arbitrarily small, the FLR effects must be taken into account to regularize the solution. The corresponding mode equation is the exact analogue of that describing shear Alfvén wave mode conversion to KAW and thus, describes GAM mode conversion to KGAM.

At the resonant point, where the local GAM frequency $\omega^2_\delta(r_\gamma)$ is the same as the driving frequency, we can expand $\omega^2_\delta(r) = \omega^2_0 + \partial \omega^2_\delta / \partial r (r-r_\gamma)$. We also assume that the external pump is of the form $S_0 = \delta E_0 \exp(-i\omega_0 t)$. Let $r-r_\gamma = \xi z$ and introduce the normalized radial electric field $E' = -\delta E_0 c_i \rho^2 / (\pi S_0 \xi^2)$, where, $\kappa = |\partial \omega^2_\delta(r)/\partial r|_{r=r_\gamma}$ and $\kappa \xi^3 / (C_i \rho^2) = 1$. Then, the GAM mode equation becomes the standard Scorer’s equation,

$$\left( \frac{\partial^2}{\partial z^2} + z \right) \delta E' = -\frac{1}{\pi},$$ (19)

with the solution given in terms of the Airy functions:

$$\delta E' = c_1 A_i(-z) + c_2 B_i(-z) + G_i(-z),$$ (20)

where $A_i$ and $B_i$ are, respectively, Airy functions of the first and second kind, and $G_i$ is a function involving integrals of $A_i$ and $B_i$. Taking the proper boundary conditions, i.e., imposing only right-going wave for KGAM and regular behavior as $r \to -\infty$, one can find that $c_1 = i$ and $c_2 = 0$. In the $|z| \gg 1$ region, we get the asymptotic solution for $\delta E_r$, written as

$$\delta E_r = -\frac{\pi^{1/2} S_0}{(\kappa^2 C_i \rho^2)^{3/4}} \left( \frac{\xi}{r-r_\gamma} \right)^{1/4} \times \exp \left\{ i \frac{2}{3} \left( \frac{r-r_\gamma}{\xi} \right)^{3/2} + \frac{\pi}{4} \right\} + \frac{S_0}{\kappa (r-r_\gamma)},$$ (21)

for $r-r_\gamma > 0$; and

$$\delta E_r = \frac{S_0}{\kappa (r-r_\gamma)},$$ (22)

for $r-r_\gamma < 0$. In Eq. (21), the first term on the right hand side represents the KGAM, while the second term, which is also present on the right hand side of Eq. (22), reflects the electric field response to the external source, as given in Eq. (15). The mode conversion of the source field to short wavelength of the GAM continuum is shown in Fig. 3.

**Fig.3** Mode conversion from GAM to KGAM at the resonant point of GAM continuous spectrum. The dashed curves denote the plasma response to the external source, given in Eq. (15), while the solid curves denote the KGAM.

### 3 Energetic-particle induced GAM: linear excitation and radial mode structure

Energetic particle physics are readily included in Eq. (5), since it coincides with the flux-surface-averaged vorticity equation. This, in turn, can be taken in a reduced form of Eq. (13) in Ref. [16], which is appropriate for investigating fast ion excitations of low frequency waves. As EGAM is driven by wave-particle resonant interactions of GAM fluctuations with the transit motion of EPs, we require that the characteristic temperature ratio of EPs to bulk ions to be $T_e/T_i \sim 9^2$. Due to the zonal mode structure of GAM, the parallel wavenumber is essentially zero and the wave-particle resonance condition for circulating particles is:

$$\omega = p \omega_{ti} = 0,$$ (23)

with $\omega_{ti} \equiv v_{ti}/q R_0$ the transit frequency and $p$ an integer. In the limit of small but finite magnetic drift orbit for EP, the primary transit resonance ($p=1$) is most important due to the fact that the EPs coupling to higher order transit resonances are proportional to $(k_i \rho_d)^{2p}$, with $k_i \rho_d$ the magnetic orbit normalized to the radial wavelength, and thus is small.

Due to the mode structure of $n=0$ and $m \approx 0$, GAM can not be driven by the expansion free energy of EPs and effective linear mode excitation requires a velocity space anisotropy in the equilibrium EP distribution function \cite{Lange1, Hnatow1, Hnatow2, Hnatow3, Hnatow4}. For a single pitch-angle slowing-down equilibrium EP distribution function, i.e., $F_{0h} = c_0(r)\delta(\Lambda - \Lambda_0)H_E$, where $\delta(x)$ is
the Dirac delta function, $\Lambda \equiv \mu/E$ is the pitch angle, $\mu = v^2/(2B)$ is the magnetic moment, $c_0(r) = \sqrt{2(1 - \Lambda^2 B)n_b(r)/(4\pi B L n_b(E_c/E_l))}$, $n_b(r)$ is the density of the EPs beam, $E_b$ and $E_c$ are, respectively, the EP birth and critical energies, and $H_E = \Theta(1 - E/E_b)/(E^{5/2} + E_c^{3/2})$, with $\Theta(1 - E/E_b)$ being the Heaviside step function, the local dispersion relation of EGAM in the small EP drift orbit limit from the surface averaged quasi-neutrality condition can be obtained as [18]

$$E_{EGAM} = -1 + \frac{\omega^2}{\omega_c^2} + N_b$$

$$\times \left[ C \ln \left( 1 - \frac{\omega^2_{tr,b}}{\omega_c^2} \right) + \frac{D \omega^2_{tr,b}/\omega_c^2}{1 - \omega^2_{tr,b}/\omega_c^2} \right] = 0; \quad \text{(24)}$$

where $\omega^2_{tr,b} = \sqrt{2E_0(1 - \Lambda^2 B)/(qR_0)$, $N_b = \sqrt{1 - \Lambda^2 B^2 n_b/(4\ln (E_b/E_c) n_c)$, $C = (2 - \Lambda B)(-2 + 5\Lambda B)/(2(1 - \Lambda^2 B)^{3/2})$ and $D = \Lambda B(2 - \Lambda B)^2/(1 - \Lambda^2 B)^{5/2}$.

Note that, in Eq. (24), the EP transit resonance drive comes only from the first term of EP response (the logarithmic term), while the second term (in the square bracket) determines how much the real frequency of local EGAM is shifted downward with respect to the local GAM continuum frequency. From Eq. (24), EGAM excited by transit resonances of EPs is locally unstable only when $C > 0$, which gives the necessary condition for local instability

$$\Lambda_B > 2/5. \quad \text{(25)}$$

In the large aspect ratio tokamak approximation, EPs are trapped only as $\Lambda_B > 1 - \epsilon$, which sets the upper limit of $\Lambda_B$ for the validity of our theory. It is noted, here, that the local EGAM dispersion relation, Eq. (24), is very similar to that of beam-plasma instability, and has two unstable branches [4]. When $\omega_G < \omega_{tr,b}$, it is the GAM branch with a real frequency very close to the local GAM continuum frequency; when $\omega_G > \omega_{tr,b}$, we have the beam branch, with its real frequency very close to $\omega_{tr,b}$, i.e., determined by EPs. In this paper, we will focus on the beam branch; an EP-mode indeed. From here, it is seen that EGAM, i.e., the EP branch, with its frequency lower than the local GAM continuum frequency, is generally not degenerate with GAM. This is one of the major differences with the beam-plasma instability (see section 5).

### 3.1 Radial mode structure of EGAM

Radial non-uniformities of both the EP source as well as GAM continuous spectrum determine both mode structures and threshold conditions. The dependence of the EGAM radial structure and nonlocal properties on the EP density profile and/or GAM continuum is discussed in recent works [4,7,18,19]. For the convenience of discussion, we introduce here two important scale lengths in the global properties of GAM/EGAM problem: the characteristic scale length of GAM continuous spectrum $L_G \equiv |\omega_{bG}(r)/(\partial \omega_{bG}(r)/\partial r)|$ and the scale length of EP density profile $L_E$.

In Ref. [18], EGAM driven by a sharply radially localized EP source is considered and a small EP drift orbit is assumed, consistent with $L_E \ll L_G$ and with the fact that EPs are unimportant, at the mode conversion layer to KGAM. Both thermal ions and EPs are treated by kinetic theories. A bounded solution of EGAM is found, radially trapped by the localization

![Fig.4 Potential well $-Q(r)$ as a function of $r/L_b$](#)
of EP drive, with a characteristic scale length given by the geometric mean of \( L_E \) and the EP drift orbit width \( L_G \). When the FOW and FLR effects of thermal ions are properly treated in the region where EPs fade away, additional physics can be found; i.e., the bounded EGAM can tunnel-couple to the outward propagating KGAM outside the localization domain of EP, resulting in a finite drive threshold of the EP density due to the continuum damping/mode conversion of EGAM to KGAM. This instability threshold of global EGAM is found to depend on the relative ratio of \( L_E \) to \( L_G \). Once again, this work is self-consistent in that \( L_E \ll L_G \) is assumed, such that the EP density is negligible at the resonant position with the GAM continuous spectrum. Strictly speaking, studies on global EGAM still analyze localized modes, in that they assume \( J_r = 0 \) instead of the surface averaged quasi-neutrality condition \( \partial_r J_r = 0 \) as the governing equation for EGAM (see, respectively, Eq. (14) of Ref. [4] and Eq. (23) of Ref. [18]). Thus, the EGAM mode equations are solved as homogeneous differential equations in both works, neglecting the coupling to global boundary conditions. This is of course consistent with the exponentially small EGAM coupling to the GAM continuum (proportional to the mode amplitude at the coupling position) due to the small drift orbit limit and localized EP profile assumption used in Ref. [18]. However, the small drift orbit limit will break down when radially broad EP drive is considered, which is, however, the condition that is relevant to experiments, as emphasized above. When the scale length of the EP density profile is of the order of or larger than that of the GAM continuum, assuming a finite EP drift orbit width compared to the mode wavelength is more appropriate. In the large EP drift orbit limit, EPs will respond adiabatically to the perturbed fields and, thus, can not contribute to the inertial layer physics when the drive orbit width is larger compared with the mode scale-length. Thus, the inertial layer contribution comes only from thermal plasma. In the limit where the scale length of the EP density profile is much larger than that of the GAM continuum, the mode equation will generally have a nonhomogeneous term from the ideal region contribution of EPs, as shown in Refs. [7,20]. This mode equation will be singular in the kinetic layer region, where the mode is resonant with the GAM continuum spectrum, and will, thus, describe the continuum damping of EGAM due to the strong coupling to the GAM continuous spectrum. In this case, EGAM can only be driven unstable when the EP resonant drive exceeds the GAM continuum damping, with a physical picture in an exact analogy with the energetic particle mode (EPM). For an arbitrary normalized drift orbit width of EPs, the corresponding EGAM mode equation becomes an integro-differential equation, which is difficult to solve. From the EGAM mode equation in both the small and extremely large EP drift orbit limit, by using the Padé’s approximation, a model differential equation can be built up that recovers both the small and large EP drift orbit limit; while in the order of the unity EP drift orbit, it can also qualitatively describe the response of EPs. This model equation, thus, consists of the physics of EGAM finite coupling to GAM continuum and mode conversion to KGAM if FLR/FOW of bulk ions is included. The detailed discussion of the EGAM mode structure and excitation threshold is beyond the scope of this work. In this work, we briefly summarize recent results on EGAM excitation by a localized EP source as an example to illustrate the radial mode structure of EGAM and to elucidate the nonlocal properties of EGAM when the GAM continuum spectrum is taken into account, showing the difference with the beam-plasma instability problem. 3.2 Radial structure of EGAM: localized drive with exponentially small coupling to GAM continuum Considering the exponentially small EGAM coupling to the GAM continuous spectrum and small EP drift orbit size as assumed, we take an EP beam localized about \( r = r_b \), where the local GAM continuum frequency is higher than the beam transit frequency, and the beam characteristic spatial scale length \( L_b \) is much shorter than that of the GAM continuous spectrum, such that the beam is localized away from the GAM resonance layer \( r_c \), where \( \omega_G(r_c) = \omega \approx \omega_{tr,b} \). In our orderings, both finite orbit width (FOW) and FLR effects are dominated by EPs inside the localization domain of the beam, while the FOW/FLR effects of bulk ions take over away from the beam radial position. In the inner \( |r - r_b| \leq L_b \) region, the bound state of EGAM can be obtained with the localized EP radial profiles. The mode scale length is set by the geometric mean of the EP drift orbit and the scale length of the EP radial profile while the radial mode structure is localized where the drive is strongest. Different eigenstates possess close real frequencies, while their growth rates decrease with the eigenmode number due to the sharp localization of the EP profile. In the outer \( |r - r_b| \gg L_b \) region, where EPs fade away, contributions from EPs are negligible and the dispersiveness is dominated by thermal ions. For this reason, we need to keep the thermal ion FLR/FOW effects to include the physics of collisionless dissipation, due to nonlocal couplings. The corresponding eigenmode equation of EGAM/GAM describes KGAM propagating in the \( r > r_c \) region, with a mode structure near \( r = r_c \) given by Airy Functions. The scale length of KGAM is readily shown to be given by \( L \approx r_b^{2/3} L_E^{1/3} \). In the following, we apply a WKB analysis and match solutions of the inner and outer region, deriving the global eigenmode dispersion relation of the nonlocal EGAM driven by sharply localized EPs.
and noting that the typical scale length of $\delta E_r$ is

$$L \approx \sqrt{\rho_{d,b} L_b} \ll L_g, \text{ with } \rho_{d,b} \text{ the beam ion drift orbit width at the birth speed, we can then get the following GAM/EGAM eigenmode equation as}$$

$$[\partial_r^2 + Q(r)] \delta E_r = 0, \quad (26)$$

where $Q(r) = [2(E_{\text{EGAM}} + F + F_3)]/[-\rho_{d,b} N_0(r) H - \rho_{d,b} G]$. The definition and physical meaning of the functions $F, F_3, H$ and $G$ are given in Ref. [15]. Here, for the sake of simplicity, we reproduce the qualitative features of $Q(r)$, illustrating the potential well $(-Q(r))$ as well as the positions of the turning points (zeros of $Q(r)$) in Fig. 4, with $T_1$ and $T_2$ the radial positions of the regular turning point pair, due to the EP radial source profile and defining the EGAM localization region, $T_2$ being finally the radial position of GAM continuum resonant excitation and of EGAM mode conversion to KGAM (see section 2.2). Eq. (26) has the WKB solution

$$\delta E_r = \left(A_1 \exp(i \int \sqrt{Q(r)} dr) + B_1 \exp(-i \int \sqrt{Q(r)} dr) \right)/Q^{1/4}(r). \quad (27)$$

The corresponding WKB dispersion relation of the eigenmode Eq. (26) can then be straightforwardly derived via asymptotic matching of WKB solutions, Eq. (27), across the turning points and is given approximately by

$$W_1 = (l + 1/2) \pi - ic^{2iW_2}, \quad l = 0, 1, 2, \ldots, \quad (28)$$

when the tunneling coefficient $c^{2iW_2}$ is formally exponentially small, with $W_1 = \int_{T_1}^{T_2} \sqrt{Q(r)} dr$ and $W_2 = \int_{T_2}^{T_3} \sqrt{Q(r)} dr$.

Eq. (28) is the well-known Bohr-Sommerfeld quantization condition, including the tunneling coupling to outgoing KGAM. Near marginal stability,

$$\gamma = -W_{1i}/(\partial W_{1r}/\partial \omega_r) - e^{2iW_2}/(\partial W_{1r}/\partial \omega_r); \quad (29)$$

expressing the mode excitation when the EP resonant drive exceeds the tunneling-convective damping, whereas $\omega_r$ is solved from $W_{1r}(\omega_r) = 0$, with $W_{1r}$ and $W_{1i}$ the real and imaginary parts of $W_1$, respectively [18]. The mode structure of EGAM from the numerical solution of Eq. (26) (cf. Fig. 5), as predicted from analytical treatment, exhibits mode localization around the radial position of the strongest EP drive and an exponentially small tunneling of the electric field to an outward propagating KGAM at the resonant layer with the GAM continuous spectrum, which is very similar to the DIII-D observations by NAZIKIAN et al. [25]. The EGAM threshold condition, due to non-local coupling to KGAM, is expected to increase with the decrease in $L_g$ and is shown numerically in Fig. 3 of Ref. [18].

![Fig. 5](image-url) Global mode structure of EGAM in the case of exponentially small EGAM tunneling coupling to KGAM. $\omega_G(r_b) \approx 1.1 \omega_{1r,b}$. We note, here, that the analysis for the EGAM, driven by localized EP beam, can be readily applied to explain the observation of EGAM at the center of a reversed shear plasma. It is $N_b \propto n_{0q}^2/n_c$, defined following Eq. (24), that really determines the localization of EP drive and, consequently, the localization of EGAM. In the reversed shear plasma, there will be a local maximum of $q$ at the center of the toroidal plasma, so there will be a local maximum of $N_b$, even if the EP radial density profile $n_0$ is radially broadly distributed, which may form a potential well leading to a localization of EGAM at the center of the plasma. Besides, the wave-particle resonance condition of GAM/EGAM with the thermal ions is $\omega = \omega_{1r,c}$, so the separation of the thermal ion transit frequency with GAM frequency $\omega_{1r,c} - \omega_Q \approx (q(T_c/T_1 + 7/4)1/2$ increases with $q$. As a result, at the center of the toroidal plasma, where $q$ has a local maximum, the ion Landau damping of GAM is minimized, resulting in a lower threshold for EGAM excitation. Thus, from our analysis, it is actually the local maximum $q$, rather than negative shear, that matters in the observation of EGAM in the reverse shear configuration, while the other profiles are the same.

### 4 EGAM saturation via wave-particle trapping

Due to the zonal structure of GAM, the mode cannot be excited by the expansion free energy of EPs. The driving mechanism must be connected with the gradients in the velocity space, provided that $\partial F_3/\partial E$ is positive. An example is given in section 3, where the drive of EGAM comes from the anisotropy in the velocity space of the EP source. As EGAM grows, resonant EPs will be trapped by the mode, with a bounce frequency characterized by the square root of the mode amplitude [27]. As a result of wave-particle trapping, the positive gradient in the velocity space may be flattened due to phase mixing. When a bounce frequency of particles trapped in the wave is comparable with the linear growth rate of EGAM, i.e., resonant EPs can not effectively transfer energy to the mode, the mode stops growing and reaches nonlinear saturation.
Here, for simplicity, we take the $T_0/T_1 \ll 1$ limit, such that the electric potential has only a surface averaged component, i.e., $\delta \phi = \delta \phi_{\text{dc}}(r) \exp(-i\omega t)$, and the electric field intensity has only a radial component, $\delta E_r$. Furthermore, as in the beam-plasma problem \cite{22}, we assume that the EGAM spectrum is essentially monochromatic in the early nonlinear phase, till it reaches saturation. The issue of power spectrum generation is addressed later in this section. Thus, the change in the particle energy per unit mass, $E$, can only be caused by the acceleration in the radial direction, associated with the radial magnetic drift

$$\dot{E} = (e/m) V_d \cdot \delta E_r. \quad (30)$$

Noting that $E = (v_\parallel^2 + v_\perp^2)/2$ and the magnetic moment $\mu$ is conserved, the rate of change of $v_\parallel$ can be expressed as

$$v_\parallel \dot{v}_\parallel = \dot{E} - \mu (\dot{\mathbf{R}} \cdot \nabla) B, \quad (31)$$

where $\dot{\mathbf{R}} = v_\| B/\|B + V_d + \delta V_E$, with $V_d$ and $\delta V_E$ the magnetic drift and the $\delta E_r \times B$ drift, respectively. The leading-order change of $v_\|_0$ is caused by the magnetic field variation along the particle trajectory and the work of the radial electric field on the radial magnetic curvature drift

$$\dot{v}_\| = -\mu \dot{\mathbf{R}} \cdot \mathbf{B} + \frac{e}{m v_\|} \dot{V}_{\text{dc}} \sin \theta \delta E_r, \quad (32)$$

where $\dot{V}_{\text{dc}}$ is the magnetic curvature drift. Assuming a beam distribution of (well) passing particles, Eq. (32), averaged over the fast time scale, yields

$$\dot{v}_\| = \frac{e}{2mv_\|} \dot{V}_{\text{dc}} \delta E_r \sin \Theta, \quad (33)$$

where $\Theta = \Theta - \omega t + k_r r$ is the phase of the resonant particles in the wave frame which is slowly varying for resonant particles. The right hand side of Eq. (33) represents the change in $v_\|_0$ due to the work done by the radial electric field when the particle is undergoing radial magnetic drift associated with the geodesic curvature, which is a parallel nonlinearity effect. Noting $\dot{\Theta} = \dot{\Theta} - \omega \dot{r} + k_r \dot{r}$ and $\dot{\Theta} = \omega_t + \delta V_E/r$, we get

$$\dot{\Theta} = -\dot{\omega}_t + \delta V_E/r + k_r \dot{r}. \quad (34)$$

Averaging over the fast time scale, and applying Eq. (32), with $v_\|_{00}$ the particle velocity matching the resonance condition, we get

$$\dot{\Theta} = -\frac{e}{2mv_\|_{00} q_0} \dot{V}_{\text{dc}} \delta E_r \sin \Theta. \quad (35)$$

This is the nonlinear pendulum equation, describing resonant particles being trapped by and exchanging energy with the wave field. When the bounce frequency of deeply trapped particles, defined as $\omega_b^2 \equiv e \dot{V}_{\text{dc}} \delta E_r/(2mv_\|_{00} q_0 R_0)$, is comparable with the linear EGAM growth rate, the mode enters the nonlinear dynamic phase and eventually saturates.

5 Analogies between GAM/EGAM and the beam-plasma system

As anticipated in sections 3 and 4, the excitation of GAM by EPs is similar to the beam-plasma problem, which is well understood since the pioneering work dating back to the 1960s and 70s, where the interaction of a supra-thermal electron beam with a 1D plasma was studied in a strong axial magnetic field. The physics of GAM/EGAM, however, also have its own peculiar features that are different from the problem of beam-plasma instability. In fact:

a. The GAM/EGAM problem is of a one-to-one analogy to the beam-plasma instability in that GAM
corresponds to the Langmuir wave, while EGAM (the EP branch of Eq. (24), is similar to the beam mode and is excited by EPs at the characteristic frequency of EP motions in toroidal geometries:

b. The beam-plasma system is most unstable when the beam-mode is nearly degenerate with the Langmuir wave. Given the velocity of the beam, the beam-mode automatically selects the parallel wavenumber that satisfies the wave-particle resonant condition and, thus, maximizes the mode growth rate. In the EGAM case, meanwhile, the parallel wavenumber is set by the toroidal geometry, in our case given by $1/(q R_0)$. Thus, for a given EP source and the corresponding transit frequency, the most unstable EGAM is not necessarily degenerate with GAM;

c. The beam-plasma system is a strictly 1D system, due to the strong axial magnetic field, which suppresses perpendicular dynamics and gives to perpendicular local structures a secondary importance. On the contrary, in the GAM/EGAM problem, as pointed out by different authors, there are EGAM modes radially trapped by the EP source profile. In fact, the EGAM radial structure can be affected not only by the nonuniformity of EPs, but also by the nonuniformity of the GAM continuous spectrum. It is shown that in the case of EGAM driven by a localized EP source, when the kinetic effect of thermal ions is properly treated, the coupling of EGAM to the GAM continuum leads to radial convective damping of EGAM. As the more general problem of EGAM excitation by a broad radial EP source is considered, EGAM can be damped due to a strong coupling with GAM continuum, with a physical picture in exact analogy with the energetic particle mode (EPM) case. Both the radial mode structure of GAM continuum and radial EGAM mode structure are expected to have important consequences on the plasma dynamics;

d. In the beam-plasma problem, the nonlinearity due to the thermal plasma response is ignorable with respect to the beam nonlinearity, when a tenuous beam is considered, and, thus, the background thermal plasma can be viewed as a linear dielectric medium, while the nonlinear evolution of the beam, due to wave-particle trapping, generates higher order harmonics, as discussed at the end of section 4. In the GAM/EGAM problem, the thermal plasma can also be thought as a linear dielectric medium, because the parallel nonlinearity, often ignored as a higher order nonlinear term, cancels out the convective perpendicular nonlinearity in the lowest order as the GAM-GAM nonlinear interaction is considered. Thus, to explain the observed EGAM’s second harmonic, we must consider the EPs nonlinearity due to wave-particle trapping (see section 4). This, of course, also provides a possible physical picture for EGAM nonlinear saturation.

e. The similarity between the GAM/EGAM and the beam-plasma problem is also evident in the perpendicular transport properties of supra-thermal particles. In both cases, the characteristic non-linear radial displacement of particle trajectories in the finite amplitude wave scales as the particle Larmor radius. More precisely, in the GAM/EGAM case, this would be the magnetic drift/bounce orbit width, as predicted from the conservation of canonical angular momentum. Energetic particle losses due to EGAM excitation will thus be due to pitch angle scattering of counter passing particles into unconfined banana orbits. However, while the beam-plasma problem can be considered strictly as a 1D case, the GAM/EGAM problem is only “nearly 1D”, due to the fundamental roles played by the radial mode structure, as discussed at point (c).

Note that, despite the strong analogies between the GAM/EGAM and the beam-plasma problem, significant differences due to the toroidal geometry and plasma non-uniformity are expected to play an important role and significantly diversify the nonlinear dynamics in both cases, in addition to that the dominant underlying nonlinear process is wave-particle trapping.

6 Conclusions and discussion

In this paper, the physics of the energetic-particle induced geodesic acoustic mode (EGAM), its radial mode structure, linear excitation and nonlinear saturation due to wave particle trapping are studied. The GAM/EGAM problem is analyzed emphasizing the similarities and differences with the well understood beam-plasma instability problem. In section 2, it is showed that the frequency of GAM constitutes a continuous spectrum, and the rich physics associated with the continuous spectrum such as the mode conversion to KGAM, are analyzed. In section 3, the linear excitation of EGAM is analyzed and the linear dispersion relation is obtained, in a form very similar to the beam-plasma problem. Like the dispersion relation for beam-plasma instability, the EGAM dispersion relation also has two unstable branches, i.e., GAM branch and EP branch (EGAM). However, the EGAM frequency is not necessarily degenerate with the GAM frequency, due to the fact that the parallel wavenumber is determined by the geometry of the toroidal plasma, unlike the case of the beam-plasma system, where the most unstable beam mode is generally degenerate with the Langmuir wave. It is also showed, by the simple example in which EGAM is driven by a radially localized EP source, that, unlike the beam-plasma system, which is strictly one dimensional, EGAM is with a radial mode structure due to the EP equilibrium profile as well as the GAM continuum due to the plasma nonuniformity. EGAM non-linear saturation via wave-particle trapping is studied in section 4, and the possibility of EGAM’s second harmonic generation due to wave-particle trapping is also discussed. In section 5 the essential similarities and differences of GAM/EGAM and the well known beam-plasma problem are analyzed. More detailed analysis of these important issues will be presented in another paper.
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