Spontaneous excitation of geodesic acoustic mode by toroidal Alfvén eigenmodes

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Abstract - Spontaneous nonlinear excitation of geodesic acoustic mode (GAM) by toroidal Alfvén eigenmodes (TAE) is studied within the framework of gyrokinetic theory. The dispersion relation for the parametric decays of a pump TAE mode into a TAE lower sideband and a GAM is derived. It is shown that, in the ideal MHD first stability region, the condition for spontaneous excitation of GAM by TAEs is $\omega_0^2 > V_2^4/(4q^2R_0^2)$, in which, $\omega_0$ is the pump TAE real frequency, $V_4$ is the Alfvén speed, $q$ is the safety factor and $R_0$ is the torus major radius. The corresponding threshold condition is also derived and suggests the decay process as an effective saturation mechanism for TAE.

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Zonal structures (ZS), such as the low-frequency zonal flow (LFZF)\textsuperscript{[1,2]} and the finite-frequency geodesic acoustic mode (GAM)\textsuperscript{[3]}, are generally believed to play important self-regulation roles in the dynamics of microscopic drift wave (DW) type turbulences\textsuperscript{[4]}. ZS can be spontaneously excited by DW turbulences, and scatter DW turbulences into the stable short radial wavelength domain, and consequently, suppress DW turbulence and associated transport\textsuperscript{[5,6]}.

The nonlinear interaction of drift Alfvén waves with ZS has been explored less thoroughly than for DW turbulence\textsuperscript{[7,8]}, although their potential role in the nonlinear dynamics of toroidal Alfvén eigenmodes (TAE)\textsuperscript{[9]} has been observed in early gyrofluid simulations\textsuperscript{[10]} and discussed theoretically\textsuperscript{[11,12]}. More recent numerical simulation results\textsuperscript{[13]} show that ZS can be forced driven by TAE. In that case, however, the effectiveness of ZS in regulating nonlinear dynamics of TAE and, more generally, Alfvénic fluctuations, is reduced with respect to the case of ZS spontaneous excitation, as recently pointed out in ref.\textsuperscript{[11]}.

Nonlinear excitation of GAM is usually believed to be important mainly in the edge of tokamak plasmas due to 1) GAM Landau damping rate decreases with increasing safety factor $q$\textsuperscript{[14]}; and, more importantly, 2) frequency separation between GAM and DW turbulence, with frequencies typically scaling as the diamagnetic drift frequency $\omega_*$, such that three-wave resonant-coupling selection rules are usually satisfied only in the edge region due to sharper gradients; i.e., $\omega_G/\omega_* \sim L_p/(R_0 k_\perp \rho_i) \ll 1$ is satisfied only for $L_p \ll R_0$, which is usually the case only in the edge region. Here, $k_\perp$ is the perpendicular wavelength, $\rho_i$ is the ion Larmor radius, $R_0$ is the major radius of the torus and $L_p$ is the typical pressure gradient scale length. However, finite-amplitude Alfvén waves, e.g., TAE, excited by energetic particles, such as fusion alphas, could overcome the GAM threshold conditions due to collisionless ion Landau damping, and drive GAM nonlinearly even in the plasma core region, since $\omega_G/\omega_{TAE} \sim q \sqrt{\beta} \ll 1$, with $q$ being the safety factor and $\beta$ the ratio between thermal and magnetic pressures. We note that resonant excitation is more efficient, since the growth rate scales as the pump mode amplitude rather than as its square in the nonresonant excitation case. This process has possible significant consequences of practical importance, as one can expect that it may lead to improved confinement in the tokamak center. The present work is devoted to study the corresponding parametric decay instability.

In this paper, we investigate the spontaneous excitation of GAM by TAEs in a simple tokamak equilibrium.
with shifted circular magnetic flux surfaces. Two basic assumptions are used in our calculation: 1) both TAE and GAM satisfy $|k_a\rho_L| < 1$ and 2) GAM is dominated by electrostatic response. We note that, unlike the case of low-frequency ZS excitations by TAEs, where electromagnetic response dominates if trapped ion responses are properly treated [11], it is appropriate to assume electrostatic GAM, since its magneto-perturbation is typically two orders smaller [15,16]. Since GAM has a finite frequency, its excitation by TAE is most effective in the resonant excitation case, when TAEs and GAM satisfy frequency and wave number matching conditions, and the process is described by three-wave parametric decay instability [17,18].

Our treatment of the three-wave interactions among the pump TAE ($\omega_1$, $k_1$), the lower TAE sideband ($\omega_2$, $k_2$) and GAM ($\omega_3$, $k_3$) follows closely [6,8,11,19,20]. We adopt the electrostatic potential $\phi$ and the parallel vector potential $\delta A_\parallel$ as field variables. Here, following the convention of [21], $\delta \psi \equiv \omega_3 \delta A_\parallel/(ck_3)$ is taken as an alternative variable replacing $\delta A_\parallel$. We note that ideal MHD is recovered if one takes $\delta \psi = \delta \phi$. Since TAE typically has $nq \gg 1$, the following ballooning-mode decomposition in the $(r, \theta, \phi)$ field-aligned toroidal flux coordinates is assumed for TAE [22]:

$$\delta \phi_0 = A_0 e^{i(nq-m_0 \theta-w_0 t)} \sum_j e^{-ij \theta} \Phi_0(x-j)+c.c.$$  

$$\delta \phi_S = A_S e^{i(nq-m_0 \theta-w_0 t)} e^{i(k_{sd} dr-w_3 t)}$$

$$\times \sum_j e^{i j \theta} \Phi_0^*(x-j) + c.c.$$  

and

$$\delta \phi_G = A_G e^{i(k_{sd} dr-w_3 t)} + c.c.$$  

Here, $(m = m_0+j,n)$ are the poloidal and toroidal mode numbers, $m_0$ is the reference poloidal mode number, $nq(r_0) = m_0$, $q(r)$ is the safety factor, $x = nq - m_0 = nq(r-r_0)$, and $A_0$, $A_S$ and $A_G$ are the envelope amplitudes of the TAE pump, the lower TAE sideband and GAM, respectively. The same decomposition is also assumed for the TAE parallel vector potential.

The governing equations, describing the nonlinear interactions of TAE and GAM, are the quasi-neutrality condition

$$\frac{nq^2 e^2}{T_i} \left(1 + \frac{1}{T_i} \right) \delta \phi = \sum_{i} \langle qJ_k \delta H \rangle$$  

and the nonlinear vorticity equation [8,21]

$$\frac{c^2}{4\pi\omega^2} \frac{B}{B_\perp} \frac{\partial}{\partial l} \frac{\delta \psi + e^2}{T_i} (1 - J_k^2) F_0 \delta \phi$$

$$- \sum_{k} \left\{ \frac{qJ_k \delta H}{\omega_{0\delta}} \right\} = -i \frac{c}{B_\perp} \sum_{k=k'+k''} b \cdot k'' \times k'$$

$$\times \left[ \frac{k''^2 e^2}{4\pi} \frac{1}{\omega_{0\delta}} \delta \psi' \delta \psi'' \right]$$

$$+ \left\{ \langle J_k J_{k'} - J_{k''} \rangle \left( \delta \phi + \frac{i}{\omega_{0\delta}} \partial_\theta \delta \psi \right)' H_i' \right\}.$$  

where $\langle \cdots \rangle$ indicates velocity space integration and $\delta H$ is the nonadiabatic part of the perturbed particle distribution function, given by the nonlinear gyrokinetic equation [23]

$$-i \omega + v_1 \partial_\theta + i \omega_d \delta H = -i \omega \frac{q}{T} F_0 J_k \left( \delta \phi + \frac{i}{\omega} v_\parallel \partial_\theta \delta \psi \right)$$

$$- \frac{c}{B} \sum_{k=k'+k''} b \cdot k'' \times k' J_k \left( \delta \phi + \frac{i}{\omega} v_\parallel \partial_\theta \delta \psi \right)' \delta H''.$$  

Here, furthermore, $\omega_d$ is the drift frequency, $F_0$ is the equilibrium particle distribution function, $J_k = J_0(k, \rho)$ with $J_0$ being the Bessel function denoting finite Larmor radius effects, $\rho = v_\perp / \Omega$, $\Omega$ is the cyclotron frequency, $V_A = B_0/\sqrt{4\pi nm_i}$ is the Alfven speed and $l$ is the length along the equilibrium magnetic field line. On the right hand side of the nonlinear vorticity equation (2), the first term is the Maxwell stress, while the second term is the kinetic Reynolds stress [8].

For the electron response to TAE, with the real frequency $\omega_{TAE} \approx \omega_{0e} / 2qR_0$, we have $\omega_{tr,e} \gg \omega_{0e}$. Here, $\omega_{tr,e}$ denotes the electron transit frequency. Thus, the electron linear response to TAE, can be solved as

$$\delta H_{TAE}^x \approx -\frac{e}{T_i} F_0 \delta \psi.$$  

While solving for the ion response, we assume $k^2_i \rho_i^2 q_i^2 \ll 1$, $\beta \ll 1$, with $\beta$ being the ratio between thermal and magnetic pressures. Thus, $\omega \approx \omega_{TAE} \gg \omega_{tr,i} \gg \omega_{0i}$, and the ion response can be solved from the gyrokinetic equation, quasi-neutrality condition and vorticity equation order by order. Since the linear properties of TAE is not of particular interest in this work, only the lowest-order linear ion response to TAE will be given here:

$$\delta H_{TAE}^x \approx -\frac{e}{T_i} F_0 J_k \delta \phi.$$  

which will be used in deriving the nonlinear particle behavior.

The nonlinear GAM equation can then be obtained from the vorticity equation. The Reynolds stress can be treated following [11] in the long-wavelength limit, and we obtain the following nonlinear GAM equation:

$$-i \omega_G e_G A_G = -\frac{1}{2} \frac{e}{B} k_0 \rho_G k_0^3 \rho_{th}^2 \left( 1 - \frac{k_0^2 \rho_G^2 V_A^2}{\omega^2} \right) A_S A_0.$$  

Here, $e_G$ is the dielectric function of GAM, defined as

$$e_G \equiv \left\langle \left( (1 - J_k^2) F_0 \delta \phi - \sum_{k} \langle J_k \omega_{0\delta} \delta H \rangle \right) (n \delta \phi_G) \right\rangle.$$  

For the details of $e_G$ and various particle species linear response to $\delta \phi_G$, one can refer to [14] and references therein. Furthermore, $\rho_{th} = v_{th} / \Omega_i$ with
For the derivation of the parametric decay dispersion relation, we also need the nonlinear TAE sideband equations. The nonlinear particle sideband responses can be solved as

\[
-\omega_G \epsilon_G A = -\frac{c}{2B} k_0 \delta \kappa G \delta \psi^* \left( 1 - \frac{\omega^2}{4\omega_0^2} \right) A_S A_0. \tag{8}
\]

In deriving eqs. (9) and (10), the following lowest-order particle responses to GAM, have been used in the \(\omega_{tr,c} \gg \omega_G \gg \omega_{d,e} \) and \(\omega_G \gg \omega_{tr,c}, \omega_{d,i} \) limit:

\[
\delta H_{G,e}^L \simeq \frac{c}{\omega_0 B} k_0 \delta \kappa G \frac{c}{T_e} F_0 \delta \phi G \delta \psi^*, \tag{9}
\]

\[
\delta H_{G,i}^L \simeq \frac{i c}{\omega_0 B} k_0 \delta \kappa G \frac{c}{T_i} F_0 J_0 J_G \frac{k_0}{\omega_0} \delta \phi G. \tag{10}
\]

Substituting the nonlinear particle responses, eqs. (9) and (10), into the quasi-neutrality condition, and noting that the leading-order ion contributions vanish due to the \(v_i^0 \) odd parities, we then obtain the electron nonlinear corrections to the ideal MHD condition:

\[
\delta \phi = \delta \psi - i \frac{c}{B} k_0 \delta \kappa G \frac{1}{\omega_0^2} \delta \phi G. \tag{11}
\]

Combining eq. (11) with the following nonlinear vorticity equation:

\[
\frac{\partial}{\partial t} k_{S,\perp}^2 \frac{\partial}{\partial \psi_S} + \frac{\omega_0^2}{V_A^2} k_{S,\perp}^2 \delta \phi_S = i \omega_S \frac{c}{B} k_0 \delta \kappa G \left( k_{S,\perp}^2 - k_{S,\perp}^2 \right) \frac{1}{V_A^2} \delta \phi_G \tag{12}
\]

then yields the desired nonlinear TAE sideband equation

\[
\left[ \frac{\partial}{\partial t} - \frac{\omega_0^2}{V_A^2} \right] \delta \psi_S = 2i \frac{c}{B} k_0 \delta \kappa G \omega_0 k_{S,\perp}^2 \delta \phi_G. \tag{13}
\]

Performing \( \int \Phi_0(\cdots) \) on eq. (13), we get

\[
k_{S,\perp}^2 \epsilon_T \epsilon_S A_S A = 2i \frac{c}{B} k_0 \delta \kappa G \omega_0 k_{S,\perp}^2 A_0^2 A_G, \tag{14}
\]

where \([11]\)

\[
\epsilon_T \epsilon_S = \frac{A_0^2}{\epsilon_0 \omega^2} A_T(\omega) D(\omega, k_G), \tag{15}
\]

\[
D(\omega, k_G) = \left( \Lambda_T(\omega) - \delta \bar{W}(\omega, k_G) \right), \tag{16}
\]

with \(\Lambda_T = \sqrt{-1 - \Gamma_T^2}, \Gamma_T = (\omega^2 / \omega_A^2 - 1/4) + \epsilon_0 \omega^2 / \omega_A^2 \) and \(\delta \bar{W}(\omega, k_G, \omega)\) the role of a normalized potential energy [24]. Furthermore, \(\epsilon_0 = 2(\epsilon + \Delta' / k_0^2) \) with \(\epsilon = r / R_0\) and \(\Delta'\) the Shafranov shift in the shifted circular magnetic flux surfaces tokamak case we consider here. Solutions of \(D(\omega, k_G) = 0\) are \(\omega = \omega_T(k_G)\), with the pump TAE frequency given by \(\omega_0 = \omega_T(k_G = 0)\).

The nonlinear dispersion relation of the parametric instability can be obtained by combining eqs. (8) and (14)

\[
\epsilon_G \epsilon_T \epsilon_S = \left( \frac{c}{B} k_0 \delta \kappa G \rho_i \right)^2 k_{S,\perp}^2 \omega_0 \frac{\omega_0^2}{\omega_A^2} \omega_A^2 \left( 1 - \frac{\omega_0^2}{4\omega_0^2} \right) |A_0|^2. \tag{17}
\]

In the case when both GAM and TAE sideband are weakly damped normal modes, the dielectric functions \(\epsilon_G\) and \(\epsilon_T, \epsilon_S\) can be expanded, assuming the resonant decay frequency matching condition \(\omega_G + \omega_T(k_G) = \omega_0\):

\[
D(\omega_S, k_G) \simeq -i \frac{\partial D}{\partial \omega_0}(\gamma + \gamma_S), \tag{18}
\]

\[
\epsilon_G \simeq \frac{i}{\omega_G} k_{S,\perp}^2 \rho_i (\gamma + \gamma_G), \tag{19}
\]

in which, \(\gamma_S\) and \(\gamma_G\) are, respectively, the collisionless damping rates of TAE sideband and GAM. We thus obtain the dispersion relation of the parametric decay process

\[
(\gamma + \gamma_S)(\gamma + \gamma_G) = \Gamma_D^2, \tag{20}
\]

where the driving term \(\Gamma_D\) is defined as

\[
\Gamma_D^2 \equiv \left( \frac{c}{B} k_0 \delta \kappa G \right)^2 k_{S,\perp}^2 \omega_0^3 \frac{\epsilon_0 \omega^3}{k_{S,\perp}^2 \omega_A^2 A_T(\omega)} \times \left| A_0 \right|^2 \frac{1 - \omega_0^2}{4\omega_0^2}. \tag{21}
\]

In the ideal MHD first stability region for ideal ballooning modes [24], we generally have \(\omega_0 \partial D / \partial \omega_0 > 0\). Thus, one can predict from eq. (20) that the necessary condition for parametric instability is

\[
\omega_0^2 < \frac{\omega_A^2}{4} > 0, \tag{22}
\]

for \(\Gamma_D^2\) to be positive, i.e., the nonlinear excitation of GAM is possible only when the pump TAE frequency lies inside the upper half of the TAE gap. We note also that, in the ideal MHD second stability region for ideal MHD modes, which may be of interest for high-performance burning plasma operations, TAE modes generally have \(\omega_0 \partial D / \partial \omega_0 < 0\), then the unstable condition for the parametric instability is \(\omega_0^2 < \omega_A^2 / 4\).
Equation (20), furthermore, indicates that the pump TAE needs to overcome the damping of GAM and TAE sideband, i.e., $\rho_0^4 > \gamma G \gamma S$. Noting that, for linear TAE, we typically have [11]
\[
\frac{\varepsilon_{0\omega_0^2/\omega_A^2}}{A_T(\omega_0)} \frac{2\omega_0/\omega_A^2}{\partial D/\partial \omega_0} \sim 1;
\]
(23)
the threshold condition for the parametric decay instability can then be estimated as
\[
\left(\frac{B_0}{B_0}\right)^2 \frac{k_{0,\perp}^2}{k_{S,\perp}^2} |A_0|^2 \left(1 - \frac{\omega_A^2}{4\varepsilon_0^2}\right) \simeq \gamma S \gamma G.
\]
(24)
Noting that $A_0$ is the amplitude of the pump electrostatic potential, $\delta \phi_0 \simeq \delta \psi_0$ and $\delta \psi_0 = \omega_0 \delta A_0 / (\epsilon k_0 ||)$, we can express the threshold condition in terms of the magnetic field of the pump TAE, i.e.,
\[
\left(\frac{\delta B_0}{B_0}\right)^2_{\text{threshold}} \simeq \gamma S \gamma G \frac{k_{0,\perp}^2}{k_{S,\perp}^2} \frac{1}{4\varepsilon_0^2 R_0^2 k_{G}^2}.
\]
(25)
Here, we have noted $1 - \omega_A^2/(4\varepsilon_0^2) \sim \varepsilon_0$. Equation (25) yields, for some typical tokamak parameters, $|\delta B_0 / B_0|_{\text{threshold}} \sim O(10^{-9} - 10^{-8})$, which is comparable to that of the spontaneous excitation of ZS by TAE [11], and, thus, suggests that both nonlinear processes could be operative and provide effective saturation mechanisms for TAE instabilities.

While the present results are derived assuming $k_{G}^2 \rho_i^2 < 1$, generalization to the short-wavelength ($k_{G}^2 \rho_i^2 \sim O(1)$) regime can be readily carried out via the nonlinear gyrokinetic equation [25]. Another assumption we applied in this paper is that the frequency of TAE decreases with increasing $k_G$. This is generally the case, but more discussions are needed to assess TAE frequency behaviors vs. $k_G$ in various plasma equilibria, such as advanced scenarios; and for various plasma parameters, such as magnetic shear. In the case when GAM is heavily damped and becomes a quasi-mode, parametric instability mechanism becomes a nonresonant decay. That is, the pump TAE, in this case, is scattered into a lower-frequency TAE sideband via nonlinear ion Landau damping. Detailed investigation on this nonresonant decay process will be reported in a future publication.

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