Explaining the size scaling of confinement properties in magnetized plasmas is one of the crucial and challenging problems of fusion energy research. It has been pointed out that turbulence spreading is responsible for local turbulence intensity dependence on global equilibrium properties [1], i.e., the system size, and, thus, for the size scaling of turbulent transport coefficients. Therefore, the nonlocal character of turbulent intensity plays a crucial role in the breakdown of gyro-Bohm scaling of turbulent transport and transition to Bohm scaling, as observed in several numerical simulations [2–4].

The radial propagation of drift-wave (DW) turbulence in tokamak plasmas was first investigated by Garbet [5], in the absence of zonal flow (ZF). Turbulence spreading was investigated also in Refs. [6,7]. Later on, using a single model equation for the local turbulence intensity, Haehn et al. [8] considered the “minimal problem” for turbulence spreading, which is about spatiotemporal diffusive propagation of a patch of turbulence as a fluctuation front from an unstable to a stable or a weaker drive region. A mean field theory of turbulent transport has been developed and transition to Bohm scaling, as observed in several numerical simulations [2–4].

The propagation of soliton causes significant radial spreading of DW turbulence and therefore can affect transport scaling with the system size by broadening of the turbulent region. The correspondence of the present analysis with the description of DW-ZF interactions in toroidal geometry is also discussed.

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descriptions [2,20,21] when ZF induced modulations on a
given DW pump are considered with its sidebands (4-
wave). Since the total energy cascades into shorter radial
wavelengths via the coherent nonlinear DW-ZF modula-
tion interaction, the local DW envelope nonlinearly
steepens and the DW linear dispersion becomes stronger.
Time scales for nonlinear interaction and linear dispersion
eventually become comparable, showing analogies to the
Langmuir soliton problem. Coherent structures are, thus,
expected to form, such as DW-ZF solitons which will
propagate radially. Turbulence spreading may then occur
via DW-ZF soliton propagation with $x \sim t$, which is faster
than any diffusive process.

The coherent 4-wave DW-ZF modulational interaction
model for toroidal plasmas [21] has been derived based on
the first-principles nonlinear gyrokinetic equation [22]. In
this work, the same theoretical treatment is applied in a
simplified slab geometry [23], where $x$, $y$, $z$ coordinates
correspond to, respectively, the toroidal coordinates $r$, $\theta$, and $\zeta$.
In particular, the radial wave number $k_s$ may be
understood as the wave number of the DW radial envelope,
$k_s = n \Omega_s / q_d / dr$ [22,20,21]. In this respect, the results ob-
tained in the present simplified slab model [24] can be
expected to hold, at least qualitatively, in toroidal plasmas.

We start from the slab analysis of the electrostatic
DW-ZF interaction model proposed in [23]. Similar to
the Hasegawa-Mima model, using two-fluid description
and quasineutrality condition, one can straightforwardly
derive the DW evolution equation in the form [23]:

\[
(1 - \rho_s^2 \nabla^2) \partial_t \phi_d - (c_s^2 / \Omega_s) \nabla \phi_d \times \hat{z} \cdot \nabla \ln n_0
\]

\[
- (c_s^2 / \Omega_s) \nabla \phi_z \times \hat{z} \cdot \nabla \phi_d + (c_s^2 \rho_s^2 / \Omega_s) \nabla
\cdot \left[ \nabla \phi_z \times \hat{z} \cdot \nabla \phi_d + \nabla \phi_d \times \hat{z} \cdot \nabla \nabla \phi_d \right] = 0;
\]  

where $c_s = \sqrt{T_e / m_i}$, $\rho_s = c_s / \Omega_s$ is the ion Larmor radius
at the sound speed and the scalar potential is normalized to
$T_e / e$; the ZF potential $\phi_z = \langle \phi \rangle$, where $\langle \cdot \cdot \cdot \rangle$
represents flux surface averaging [$(y, z)$ plane]. The last two terms on
the right-hand side correspond to higher order Reynolds
stress corrections due to nonlinear polarization drift,
$O(k_s^4 \rho_s^4)$, which can be ignored when $|k \rho_s| \ll 1$. Mean-
while, ZF has $k_d = k_z = 0$; thus, electrons do not behave
adiabatically in the ZF potential. We can describe ZFs by
the condition of no net radial flux:

\[
\hat{d}_r \nabla^2 \phi_z - (c_s^2 / \Omega_s) \nabla \cdot \left[ \nabla \phi_d \times \hat{z} \cdot \nabla \nabla \phi_d \right] = 0.
\]  

As in Refs. [2,20,23], we consider a coherent drift wave
with single toroidal number $n$, or constant $k_s$ in slab
geometry. Thus, the 2-field coupled set of DW-ZF evolu-
tion equations are readily cast in the form

\[
(1 + k_z^2 - \partial_z^2) \partial_t \phi_d + i \omega_n(x) \omega_s(0) \phi_d = -i \frac{C}{\phi_d} \partial_z \phi_d + \text{c.c.}.
\]  

\[
\partial_t \phi_z = i C \phi_d \partial_z \phi_d + \text{c.c.};
\]

where $\omega_n = -k_c c_s / L_n$ is the diamagnetic drift frequency
and $L_n = (d \ln n / dx)^{-1}$ measures the nonuniformity scale;
$C = k_c \Omega_d / \omega_s(0)$ is a constant, while space and time have
been normalized to $\rho_s$ and $\omega_s(0)$, respectively. Note the
structural similarity of Eqs. (3) and (4) with Eq. (4) of [2] in
toroidal geometry where the most general form of the
leading order radial envelope wave-packet propagation is
given by Eq. (5) of [2]. Numerical simulations of the above
coupled system show that DW-ZF can form solitary struc-
tures, which coherently propagate with characteristic speed
(Fig. 1). These coherent structures are envelope solitons
with wavelength of the carrier wave comparable to the
envelope width, suggesting that turbulence spreading can
be caused by soliton formation due to balance between DW
dispersion and trapping by nonlinearly generated ZFs.

For the sake of simplicity, we initially ignore linear
growth or damping and dissipation of both DW and ZF.
For now, we also take $\omega_n$ constant. The $\omega_n(x)$ profile intro-
duces extra effects of finite system size, which will be
discussed elsewhere. Furthermore, we assume a coherent
DW form $\phi_d(x, t) = A_d \mu_d(x, t) \exp(ik_s x - i \omega_d t)$, in which
$A_d$ is the maximum perturbation amplitude, usually
$10^{-4} - 10^{-2}$, the normalized envelope function $u_d(x, t)$ is
chosen to be real and long-scale $[\partial u_d^2] \ll k^2_d [u_d]$, the
phase $\varphi = k_s x - \omega_d t$ describes fast oscillations in time but
not necessarily in space, $k_s$ is the radial wave number and $\omega_d$
is the DW frequency. Substituting the DW form given
above into Eqs. (3) and (4), the coupled partial differential
equations (PDEs) can be rewritten in the following non-
linear Schrödinger equation form

\[
(1 + k_d^2)(\partial_t + \nu \partial_x)u_d + (i \omega - \partial_x^2) u_d
- i \omega \lambda u_d = -i C \partial_x \phi_d, u_d.
\]

where $\nu = -2 k_s \omega / (1 + k_d^2)$, $\lambda = (1 + k_d^2) / \omega$, and
$k_d^2 = k_z^2 + k_y^2$. For constructing a stationary solution, we

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The DW amplitude $A_d$ and the dispersion and nonlinear self-trapping process. When the second order ODE clearly indicates the competition between linear dispersion and nonlinear self-trapping process. When the DW amplitude $A_d$ increases, its envelope becomes nonlinearly steeper, i.e., $\delta$ increases; meanwhile, the DW dispersion also becomes stronger and tends to inhibit the focusing process. Formally, this corresponds to equating the three coefficients of $u''_d$, $u'_d$, and $u^3_d$, i.e.,

$$\delta^2 = 1 + k^2_d - 1/\omega = C^2/(2\omega^3)(1 + k^2_k)A^2_d,$$  \hspace{1cm} (8)

The DW wave-packet frequency $\omega$ is then readily obtained from the above quadratic equation, i.e., $\omega = [1 + \sqrt{1 + 2(1 + k^2_kCA^2_d)/[2(1 + k^2_k)]}]$. Note that the right-hand side contains the nonlinear frequency shift due to finite DW turbulence amplitude. Similarly, the parameter $\delta$ can also be determined as

$$\delta = \pm \sqrt{(1 + k^2_k)/2CA_d/\omega).}$$  \hspace{1cm} (9)

This derivation is subject to our a priori assumption $\delta \ll \omega \nu^{-1}$, which guarantees that the DW oscillation $\omega$ occurs on the fastest time scale. Substituting $\delta$, $\omega$, $\nu$ as functions of $k_x$ and $A_d$, this assumption is equivalent to $k_xCA_d \ll \sqrt{(1 + k^2_k)/2\omega(k_x,A_d)}$. Finally, given Eq. (8), Eq. (7) becomes a dimensionless ODE governing the stationary envelope function $u_d$,

$$u''_d - u'_d + 2u^3_d = 0.$$  \hspace{1cm} (10)

This ODE is analogous to that of an oscillator in the so called “Sagdeev potential” $\Phi(u_d) = (-u^2_d + u^3_d)/2$, whose solution can be written as hyperbolic secant function $u_d(\xi) = \text{sech}(\xi)$, when appropriate boundary conditions are imposed, viz. $u_d \rightarrow 0$ at $|\xi| \rightarrow \infty$. Meanwhile, the ZF solution is obtained straightforwardly by integrating Eq. (6) once, such that $\phi_d(\xi) = \int 2k_xC/(\nu \delta)A_d^2 \text{sech}^2(\xi)d\xi = \mp \sqrt{(1 + k^2_k)}A_d \text{tanh}(\xi)$ which satisfies the causality constraint, i.e., $\partial_\xi \phi_d \rightarrow 0$ when $|\xi| \rightarrow \infty$ for any initially localized DW turbulence. Expressions for DW and ZF in the laboratory frame are

$$\phi_d(x,t) = A_d \text{sech} \left[ \delta \left( x + \frac{2k_x \omega}{1 + k^2_k} \right) \right] e^{i(k_x x - \omega t)}.$$  \hspace{1cm} (11)

$$\phi_c(x,t) = \sqrt{(1 + k^2_k)A_d \text{tanh} \left[ \delta \left( x + \frac{2k_x \omega}{1 + k^2_k} \right) \right]}.$$  \hspace{1cm} (12)

From Eqs. (11) and (12), it generally follows that ZF potentials have radially moving structures of hyperbolic tangent shape; meanwhile, $E_z = -\partial \phi_c/\partial_x$ manifests itself as scalar-potential wells in the background plasma and trap the corresponding DW packets. The radial scale of the soliton is $\rho_s^2/(L_d A_d)$ according to Eq. (9). Figure 1 shows the spatiotemporal evolution of two counter-propagating DW-ZF solitons, which are solutions of the original coupled PDEs, given $k_x = 0.3$, $\Omega_x/\omega = 100$. For consistency with our analytic approach, we have chosen initial $k_x$ and $A_d$ to satisfy the a priori assumption $\delta \ll \omega \nu^{-1}$. Furthermore, we assumed no $\omega$, equilibrium variation and no growth or damping and dissipation. Note that the two envelope solitons remain unchanged in both real and $k$ space after the collision, although the dynamics during the collision can be quite complicated. This is one of the soliton’s essential features.

The radial propagation velocity of DW-ZF solitary structures, $\nu$, depends on both the radial wave number and the DW amplitude. It is different from the group velocity, $\nu_g = \partial \omega_x/\partial k_x$, which is determined by $k_x$ through the linear dispersion relation only. Therefore, the solution of Eqs. (11) and (12) gives a two-parameter, $k_x$ and $A_d$, family of solitons. Figure 2 shows the relation between $\nu$ and $k_x$ for small initial amplitude $A_d = 0.003$. Numerical and analytical results agree well when $k_x \leq 0.4$. The discrepancy when $k_x > 0.4$ originates from the breaking of the a priori assumption that $\delta \partial_x \phi_d$ can be ignored in Eq. (5). Moreover, when $A_d$ increases to about $10^{-2}$, the analytical result is no longer valid either, since the ignored term $O(\nu \delta^3)$ modifies the solution at larger $k_x$ or $A_d$, according to Eq. (9).

Our numerical simulation results for $A_d \geq 0.01$ show that the dominant asymptotic ($t \rightarrow \infty$) DW turbulence behavior is still of soliton type and the propagation velocity $\nu$ increases with the DW amplitude $A_d$. We observe that the DW radial wave number no longer corresponds to its initial value but rather to $\delta$, which is mainly determined by the...
amplitude $A_d$ alone. There seems to be a transition from a two-parameter to a one-parameter family of soliton solutions of the coupled system. If the amplitude becomes even larger, e.g. $A_d \geq 0.02$, the initially localized DW-ZF soliton breaks into many pieces in the form of solitons and wave trains; similar to Gardner’s work [25] on the Korteweg–deVries equation, in which it is shown that a localized but otherwise arbitrary initial perturbation will generate a conventional wave train, quickly destroyed by dispersion, and a finite number of solitons, which characterize the asymptotic solution.

We studied the DW-ZF initial value problem in more general cases as well, i.e., in the presence of linear growth or damping, dissipations, and variation of equilibrium profiles. Figure 3 shows the evolution of DW turbulence out of initial random noise, with strong DW growth rate $\gamma_d(0) = 0.1$, uniform ZF damping rate $\gamma_z = 0.075$ and $L_p = 150 \rho_s$, which represents the system size. Dissipations are also included. The drift frequency $\omega_\perp(x)$ has Gaussian shape centered at $x = 0$; DW turbulence is linearly unstable in the central region ($|x| < 80 \rho_s$) but is damped in the outer region ($|x| > 80 \rho_s$), i.e. $0.15 \exp(-x^2/(75 \rho_s)^2) - 0.05$, while the ZF is uniformly damped. Figure 3 clearly shows formation and propagation of solitons, which however exhibit more complicated dynamic behaviors, for instance, growing amplitudes, slowing down of propagation speed, soliton breaking, turbulence bursting and more. Since coupled PDEs generally describe infinite-dimensional dynamical systems, DW turbulence dynamics appears mostly chaotic in the corresponding parameter space, $(\gamma_d, \gamma_z)$. Solitons may bounce back at their turning points, possibly enhancing nonlinear interactions inside the turbulent region and impacting the size scaling of turbulence transport. Figure 3 also demonstrates that the nonlinearly saturated turbulence has spread into a much broader region than that of its linear mode structure sampling global equilibrium properties and, again, reflecting finite system size.

In summary, we have demonstrated the novel result that coherent structures such as radial envelope solitons can be constructed self-consistently in a two-field DW-ZF model and cause significant radial turbulence spreading in a slab plasma. Horton and Meiss [26] considered poloidal DW solitons in the absence of ZF and dispersion due to polarization drift. Despite the difference in the underlying physics, the inverse scattering method and statistical approach they adopted could also be applied here to resolve the radial DW-ZF dynamics. We have also shown the structural analogy of the underlying coupled PDEs for the nonlinear evolution of the DW radial envelope and ZF amplitude with the corresponding equations derived in toroidal geometry [2,20], demonstrating the generality of the present results and the possibility of readily extending them in future works. The size scaling of DW turbulence will also be discussed in detail in a separate work.

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[24] For further clarifications on “the subtle differences between the slab and toroidal geometries”, please see Ref. [23].