On nonlinear physics of shear Alfvén waves\textsuperscript{a)}

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(Received 20 November 2012; accepted 22 January 2013; published online 14 May 2013)

Shear Alfvén waves (SAW) are electromagnetic oscillations prevalent in laboratory and nature magnetized plasmas. Due to their anisotropic nature, it is well known that the linear wave propagation and dispersiveness of SAW are fundamentally affected by plasma nonuniformities and magnetic field geometries, such as the existence of continuous spectrum, spectral gaps, and discrete eigenmodes in toroidal plasmas. This work discusses the pure Alfvénic state and demonstrates the crucial roles that finite ion compressibility, non-ideal kinetic effects, and geometry play in breaking it and, thereby, the nonlinear physics of SAW wave-wave interactions. \copyright 2013 AIP Publishing LLC. [\url{http://dx.doi.org/10.1063/1.4804628}]

I. INTRODUCTION

Hydromagnetic Alfvén waves, discovered by Alfvén,\textsuperscript{1} are fundamental low-frequency electromagnetic oscillations in magnetized plasmas. Alfvén waves are found to be prevalent in nature and laboratory plasmas. For example, Alfvén waves are often found to be excited by energetic charged particles, produced by either geomagnetic storms or plasma heating, in space and fusion plasmas. Alfvén waves, manifested as geomagnetic oscillations, are also often observed within the Earth’s magnetosphere, due to solar wind perturbations. Since Alfvén waves carry electromagnetic perturbations, charged particles can exchange energy and momentum with the waves, resulting in acceleration, heating, and transports across the confining magnetic field. Alfvén waves, thus, have been proposed as the mechanism for the solar corona heating.\textsuperscript{2} Meanwhile, rapid loss of energetic/alpha particles due to Alfvén wave instabilities is a major issue of concern in burning fusion plasmas, such as the International Tokamak Experimental Reactor (ITER).\textsuperscript{3,4}

In uniform plasmas, it is well known that Alfvén waves consist of the compressional and shear Alfvén waves (CAW and SAW). While SAW is anisotropic, with $\omega \approx k_i v_A$, CAW is isotropic, with $\omega \approx k v_A$. Here, $\omega$ being the wave frequency, $k$ being the wave vector, $k_i \equiv k \cdot b_0$, $v_A$ is the Alfvén speed, and $b_0 \equiv B_0 / B_0$ is the unit vector along the equilibrium confining magnetic field. Due to its lower frequency and near incompressibility, SAW is generally easier to be excited than CAW. Furthermore, since, for SAW, the wave group velocity is along $B_0$, $v_g \approx v_A b_0$, and energetic/alpha charged particles also move mainly along $B_0$ with velocities comparable to $v_A$, it is, in general, easier for SAW than CAW to satisfy the wave-particle resonance condition. SAW has thus been the focus of extensive research in both space and fusion plasmas and will also be the focus of the present paper.

In nonuniform plasmas, e.g., tokamaks, radial nonuniformities render SAW frequency radially varying, i.e.,

\begin{equation}
\omega^2 = k_i^2(r) v_A^2(r) \equiv \omega_A^2(r) \text{ and SAW spectrum becomes a continuum. In terms of initial perturbations, fluctuations such as fluid displacement, } \delta \xi, \text{ then behave time asymptotically as }
|\delta \xi| \propto \exp[-i\omega_A(r)t].
\end{equation}

Equation (1) indicates that the corresponding radial wave number

\begin{equation}
|k_i| \equiv |d\delta \xi/dr|/|\delta \xi| \propto \left|\omega_A^2(r)\right|t,
\end{equation}

\text{i.e., perturbations become “singular” as } t \rightarrow \infty. \text{ This “singular” behavior is, of course, consistent with the existence of SAW resonant absorption at steady state\textsuperscript{5,6} and the physics of linear mode conversion to the kinetic Alfvén wave (KAW)\textsuperscript{1} with the following dispersion relation, for } b_i = k_i^2 v_A^2 < 1:

\begin{equation}
\omega^2 = \omega_A^2[1 + b_i(T_i/T_e + 3/4)].
\end{equation}

In realistic fusion devices, e.g., axisymmetric tokamak plasmas, there are, in addition to the radial nonuniformities, the poloidal asymmetries. A SAW wave packet propagating along a given magnetic field line will, thus, experience periodic variations, i.e., a lattice symmetry, which then leads to the appearance of “gaps” in the SAW continuum.\textsuperscript{8} Additional equilibrium variations, meanwhile, constitute as “defects” to this lattice\textsuperscript{9} and, thereby, introduce discrete localized bound states, known as Alfvén Eigenmodes within the SAW gaps.\textsuperscript{10} Thus, in realistic fusion plasmas, SAW consists of both continuous and discrete spectra, which are of fundamental importance to linear as well as nonlinear SAW dynamics.

In fusion devices, the plasma typically consists of a thermal and an energetic component, produced by external heating or fusion reactions. These energetic charged particles (EP) have dynamics frequencies, typically, in the range of SAW frequency and, thus, render SAW-EP resonance feasible. In addition, EPs have finite density and pressure gradients, which provide the necessary free energy to excite
SAW instabilities if there is finite force acting on the resonant EPs. Since SAW frequency is much lower than the cyclotron frequency, the EPs mainly experience gyrophase-averaged force due to finite \( \delta E_{\parallel} \), \( \delta B_{\parallel} \) and \( (v_d \times \delta B_{\perp}) \cdot b_0 \), where \( \delta E = \delta E_{\parallel} \cdot b_0 \), \( \delta B = \delta B_{\parallel} \cdot b_0 \). \( v_d \) denotes the \( \mathbf{V} \)B and \( \mathbf{k} = b_0 \cdot \mathbf{V} \)b (magnetic curvature) drift, and \( \delta \mathbf{E} \) and \( \delta \mathbf{B} \) are perturbed wave fields.

The existence of SAW continuum indicates that SAW instabilities in fusion plasmas generally consist of singular inertial and regular ideal regions (in terms of ideal magnetohydrodynamic (MHD) terminology). Asymptotically matching solutions in these two regions then leads to a dispersion relation of the following rather general form:11–17

\[
i \Lambda(\omega) = \delta \tilde{W}_f + \delta \tilde{W}_k(\omega) . \tag{4}\]

In Eq. (4), \( \Lambda(\omega) \) corresponds to the generalized inertia, while \( \delta \tilde{W}_f \) and \( \delta \tilde{W}_k(\omega) \) correspond to the generalized potential energy due to thermal plasma and EPs, respectively. Equation (4) is dubbed as the generalized fishbone dispersion relation.11–17 In terms of effects, \( \Lambda(\omega) \) determines the structure of continuum along with gaps, \( \delta \tilde{W}_f \) determines the existence of discrete AE, and \( \delta \tilde{W}_k(\omega) \) provides the instabilities mechanisms as well as additional unstable branches via the frequency dependence of \( \delta \tilde{W}_k(\omega) \). Since the existence of these new unstable modes is entirely due to EPs, they have been dubbed Energetic Particle Modes (EPM).15 The fishbone instability11 is one celebrated example of EPM.

Ultimately, we need to develop the nonlinear physics in order to properly construct the SAW turbulent spectrum and assess the associated heating/acceleration and transports. In terms of EP-driven SAW instabilities, there are two possible nonlinear routes. One corresponds to nonlinear SAW-EP interactions. Within this route, there are currently two paradigms. One is the “bump-on-tail” paradigm, which is based on wave trapping, including effects of source and dissipation. Here, the wave trapping occurs due to radial detuning of wave-particle resonance. This paradigm has been extensively developed by Berk, Breizman and coworkers,18–20 and applied to explain experimental observations [cf. Ref. 21 for a recent review]. The other paradigm will be dubbed as the “fishbone” paradigm, in which, due to frequency chirping, there is little resonance detuning and the wave-particle phase is locked.11,22 On the other hand, the duration of wave-particle interaction is limited due to the finite radial localization of the mode structures.11,22,23 This paradigm still needs further development along with feedbacks from realistic large-scale simulations, which have begun to come online. Ultimately, one can perceive the unification of these two paradigms.

The other route to of nonlinear physics is via nonlinear wave-wave interactions and the resultant spectral wave energy transfer. We will focus on this route in the present paper.

In Sec. II, we will review the unique feature of nonlinear SAW, the pure Alfvénic state, where nonlinear interactions diminish, and discuss how to break it. In Sec. III, we discuss the effects of finite ion compressibility and demonstrate that kinetic effects qualitatively and quantitatively modify the SAW parametric decay process. In Sec. IV, we focus on the nonlinear excitation of convective cells and demonstrate the importance of kinetic effects due to finite ion Larmor radius (FILR) corrections to the Reynolds stress, and that the electrostatic and magnetostatic convective cells are generally strongly coupled. Section V demonstrates the importance of geometries by examining the nonlinear excitation of zonal structures by Toroidal Alfvén Eigenmodes (TAE). Conclusions and discussions are presented in Sec. VI.
potential $\delta A$, and adopting the perturbed flux function, $\delta \psi$, given by $b_0 \cdot \nabla \delta \psi = -\partial_t (\delta A_\parallel /c)$, we have

$$\left(\frac{c^2}{4\pi}\right)(b_0 \cdot \nabla)^2 \nabla^2 \delta \psi + \partial_t (\nabla \cdot \delta J_\perp) = 0.$$  

(11)

From Eq. (7), the perturbed perpendicular current, $\delta J_\perp$, in Eq. (11) consists of two terms, i.e.,

$$\delta J_\perp = \delta J^{(1)}_\perp + \delta J^{(2)}_\perp,$$  

(12)

where $\delta J^{(1)}_\perp$ is the linear ion polarization current

$$\delta J^{(1)}_\perp = (c/B_0) b_0 \times \nabla \delta \psi,$$  

(13)

and $\delta J^{(2)}_\perp$ is the nonlinear current due to $F_p^{(2)}$, i.e.,

$$\delta J^{(2)}_\perp = -(c/B_0) b_0 \cdot F_p^{(2)}.$$  

(14)

In Eq. (13), $\delta u_\perp$ is, in the lowest order, the $E \times B$ velocity or, adopting the scalar potential $\delta \phi$ as the other field variable, we have

$$\delta u_\perp = (c/B_0) b_0 \times \nabla \delta \phi.$$  

If we now make the additional ideal MHD assumption $\delta E_\parallel = 0$ or $\delta \phi = \delta \psi$, Eq. (11) then becomes

$$c^2[(b_0 \cdot \nabla)^2 - v_A^2 \partial_t^2 + 4\pi \partial_t (\nabla \cdot \delta J^{(2)}_\perp)] = 0,$$  

(15)

where the nonlinear term $\nabla \cdot \delta J^{(2)}_\perp$ further reduces to

$$\nabla \cdot \delta J^{(2)}_\perp = -\partial_t (b_0 \cdot F_p^{(2)}).$$  

(16)

Equation (16) indicates that, for perturbations satisfying the following Walén relation:

$$\delta u_{\perp W}/v_A = \pm \delta B_{\perp W}/B_0,$$  

(17)

Re + $\text{Mx}$ = 0 and $\nabla \cdot \delta J^{(2)}_\perp = 0$. Equation (17), in terms of $\delta \phi$ and $\delta A_\parallel$, becomes

$$\delta \phi_{W}/v_A = \mp \delta A_{\perp W}/c$$  

(18)

or, taking, $\delta \phi = \delta \psi$,

$$[\partial_t \mp v_A b_0 \cdot \nabla] \delta \phi_{W} = 0,$$  

(19)

which corresponds to a SAW propagating either along or against $B_0$ and, more significantly, satisfying the nonlinear SAW equation, Eq. (15), for any arbitrary amplitude, $\delta \phi_{W}$. This is the celebrated pure Alfvénic state. In other words, within the present approximations, a large-amplitude SAW, satisfying $\omega \pm k \cdot v_A$ can exist for a long time scale without being broken by nonlinear processes. To break this pure Alfvénic state, one would need to either introduce higher-order nonlinearities and, thereby, examine dynamics at longer time scales, or, as adopted in the present work, remove the three fundamental assumptions employed here, e.g., incompressibility, ideal MHD constraint, and homogeneity/geometry.

III. FINITE ION COMPRESSION AND PARAMETRIC DECAYS

Since ion sound wave (ISW) compresses plasmas along $B_0$, we may break the pure Alfvénic state by considering parametric decays of a pump SAW, $\Omega_+ = (\omega_0, k_0)$, to a daughter SAW, $\Omega_- = (\omega_s, k_s)$, and an ISW, $\Omega_s = (\omega_r, k_r)$, where $\Omega_+ = \Omega_+ - \Omega_0$. This decay process was analyzed first by Sagdeev and Galeev, in the long-wavelength MHD limit. The result indicates that, for $\beta \ll 1$, the decay instability is a backscattering process, i.e., $\Omega_-$ is propagating counter to $\Omega_+$ along $B_0$, and the dispersion relation is given by

$$\epsilon_s \epsilon_{\perp -} = C_l \epsilon \delta \phi_{W}/(T_e c),$$  

(20)

where $\epsilon_s \simeq 1 + b_s - (k_l^2 c^2 /\omega^2)$ and $\epsilon_{\perp -} \simeq 1 - (k_l^2 c^2 /\omega^2)$ are, respectively, the dielectric constants of ISW and SAW in the MHD limit; $b_s = k_s^2 c^2 /\rho_s$, $\rho_s$ is the ion Larmor radius computed at the sound speed $c_s$; and

$$C_l = \frac{b_0}{1 + \gamma_t T_i / T_e} \cos^2 \theta.$$  

(21)

Note that $C_l \propto \cos^2 \theta$ and $\theta$ is the angle between $k_0$ and $k_\perp$ and, thus, the decay maximizes around $\theta = 0, \pi$, i.e., when $k_\perp$ is nearly parallel to $k_0$. This, as we will see later, carries important implications to the transport processes.

This parametric decay process was extended to the drift-kinetic limit and, recently, to the gyrokinetic limit. In the nonlinear gyrokinetic analysis, we consider a uniform Maxwellian plasma with $\beta \ll 1$. Again, we adopt the $\delta \phi$ and $\delta A_\parallel$ field variables, and the field equations are the quasineutrality condition, and the parallel Ampére’s law or the vorticity equation. Thus, the nonlinear gyrokinetic equations become

$$\delta f = -(q/T) F_M \partial_t (e^{-q V} \delta \phi)$$  

(22)

and

$$[\partial_t + v_A \partial_v + (\delta u_{Eg})_v \cdot \nabla] \delta \phi = -(q/T) F_M (\delta L_\parallel)_v,$$  

(23)

Here, $\delta L_\parallel = e q^2 \delta L_\parallel$, $\delta L = \delta \phi - (v_A/c) \delta A_\parallel$, $(\delta u_{Eg})_v = (c/B_0) b_0 \cdot \nabla (\delta L_\parallel)_v$ and $(...)_v$ denotes gyro-averaging of $(...)$. The corresponding parametric dispersion relation is then given by

$$\epsilon_K \epsilon_{\perp -} = C_K \epsilon \delta \phi_{W}/(T_e c)^2,$$  

(24)

where $\epsilon_K = 1 + \tau + \tau \gamma s \bar{Z} (\bar{\gamma})$ and $\epsilon_{\perp -} = \tau (1 - \Gamma_{-}) - (k_l^2 c^2 /\omega^2)$ are, respectively, the kinetic dielectric constants of ISW and SAW (KAW). Here, $\bar{Z} \simeq \omega_s / (k_s^2 c^2)$, $\bar{\gamma} \simeq \omega_s / c_s$, $v_{th}$ is the ion thermal speed, $\tau = T_e/T_i$, $\Gamma_s = I_0 (\lambda_s/e)^{-2}$, $\lambda_s = k_s^2 \rho_s^2$, $l_0$ is the modified Bessel function of the first kind, and $\Gamma_{-} = 1 + \tau (1 - \Gamma_{-}) \sim O(1)$, with $\Gamma_{-}$ defined as $\Gamma_{s}$, replacing the subscript $s$ with $-$. Meanwhile,

$$C_K = \left(\frac{\Omega_+}{\omega_0}\right)^2 \frac{b_0}{\sigma_{\perp}} H^2 \sin^2 \theta,$$  

(25)
with \( H = |\sigma_0 \sigma_\gamma - \langle J_0 J_\gamma \rangle F_{m1}\rangle / |\Gamma_\gamma| \sim O(\beta_0) \) for \(|k_\perp \rho_1| \leq O(1)\). \((\cdots)_c\) denotes velocity space integration, \(F_{m1}\) stands for the thermal ion Maxwellian distribution function, \(J_\gamma = J_\gamma(k_\perp \rho_1 / \Omega)\), \(J_\gamma\) is the Bessel function of the first kind, \(\Omega\) is the ion cyclotron frequency, and \(J_\gamma, J_\gamma\) are given by the same expressions as \(J_\gamma\) with \(s\) subscripts replaced by 0 and \(-\), respectively. Note that Eq. (25) indicates that decay process maximizes around \(\theta \simeq \pm \pi / 2\), i.e., \(k_0 \perp k_\perp\), and \(|k_\perp \rho_1| \sim O(1)\). Both predictions have recently been observed in numerical simulations.30

Equations (21) and (25) indicate that the decay instabilities are quantitatively and qualitatively different in the MHD and kinetic regimes. Specifically, we note that

\[
|C_k / C_I| \sim (\Omega / \omega_0)^2 b_0^2. \tag{26}
\]

Thus, for \(1 \geq b_0 > |\omega_0 / \Omega| \leq O(10^{-2})\), typically, we have \(C_k > C_I\) and the nonlinear decays enter into the kinetic regime. More significantly, as noted above, \(k_\perp \perp k_{01}\) in the kinetic regime, in contrast to \(k_\perp || k_{01}\) in the MHD regime. Taking, for example, \(\Omega_0\) being a radially localized Alfvénic eigenmode, such that \(k_{01} \simeq k_0 \hat{r}\), the MHD theory would then predict the decay SAW also tends to peak around \(k_\perp \simeq k_\perp \hat{r}\) and, hence, produces little transport. On the contrary, the kinetic theory would predict \(k_\perp \simeq k_\perp \hat{r}\) and, hence, significant transports.

The transition between the MHD and kinetic regimes can be understood in terms of the transition between the responsible nonlinear terms. That is, in the MHD regime, the ISW density perturbation, \(\delta n_s\), is produced by \((\delta J_0 \times B_0)\), \(b_0\) ponderative force of the SAWs, which in turn produces the nonlinear perpendicular current of the SAW. Both terms can be shown to be proportional to \(k_{01} \times k_\perp \propto \cos \theta\). On the other hand, in the kinetic regime, the parallel to \(B_0\) ponderative force, which generates \(\delta n_s\), is now dominated by the \((\delta u \cdot \nabla)\delta n_s\) convective nonlinearity, with \(\delta u\) produced by the finite \(\delta E\) of KAW. Meanwhile, the nonlinear perpendicular current is produced by the Reynolds stress involving the ISW flow, and, here, since ISW is dominantly electrostatic, the corresponding Maxwell stress is negligible. Both terms have the familiar form \((k_{01} \times k_\perp) \cdot B_0 \propto \sin \theta\). Finally, we note that Eq. (24) is derived in the resonant-decay limit, in order to highlight the major quantitative and qualitative differences. The above analyses can be readily generalized to the case of non-resonant decay (i.e., nonlinear ion Landau damping)29 and the results are qualitatively similar to the resonant-decay case.

IV. EXCITATION OF CONVECTIVE CELLS BY KINETIC ALFVEN WAVES

Another approach to break the pure Alfvénic state is via introducing the non-ideal effects, such as KAW. More specifically, here, we shall consider the nonlinear excitation of convective cells by KAW.31 Convective cells, with \(\omega_0 = 0\) and \(k \cdot b_0 = 0\), have been extensively studied in the 1970s32–35 in the context of charged particle cross-field diffusion.36 Current interest in convective cells within the fusion community is, of course, related to the observation that zonal flows/structures in tokamak plasmas may be regarded as convective cells with, however, \(k \simeq k_\perp \hat{r}\).37,38 Zonal flows/structures, meanwhile, have received extensive interest, since they may regulate the driving turbulence and, thereby, the associated transports via scattering to the shorter radial-wavelength stable domain.

While excitations of convective cells by KAW have been extensively studied in the context of generation of turbulence flows in the upper ionosphere,39,40 previous theoretical analyses often made the following two limiting assumptions. One is neglecting the FILR corrections to the Reynolds stress. The other is the decoupling between the electrostatic and the magnetostatic convective cells. Both assumptions, as will be shown, could lead to erroneous conclusions on the spontaneous excitation of convective cells by KAW.

In the present analysis, we adopt the nonlinear gyrokinetic equations and treat both electrostatic convective cell (ESCC) and magnetostatic convective cell (MSCC) on the same footing. Again, we consider a \(\beta \ll 1\), uniform Maxwellian plasma, and adopt \(\delta \phi\) and \(\delta \lambda\) or \(\delta \psi\) as the field variables. Let us consider the four-wave modulational interactions with \(\Omega_0 \equiv (\omega_0, k_0)\) being the pump KAW, \(\Omega_1 \equiv (\omega_1, k_1)\) being the convective cell mode, and \(\Omega_2 \equiv \Omega \simeq \Omega_0\) being the upper and lower KAW sidebands. From the nonlinear gyrokinetic equation, we can derive the following vorticity equation for \(\delta \psi_2\):

\[
-i \omega_2 b_\perp \delta \phi_2 = \frac{1}{2} \left( \frac{c}{B_0^2} \right) \rho_0^2 \sum_{k^2 + k'^2 = k_0^2} \Lambda_{k_0}^{\psi} (k^2 - k'^2) \times \left[ G_{k_0} G_{k_0} \delta \phi_2 \delta \phi_2 - \left( \frac{k_0 \cdot v_A}{\omega_0} \right) \delta \psi_2 \delta \psi_2 \right] \tag{27}
\]

where \(\Lambda_{k_0}^{\psi} = (k_0' \cdot k_0') \cdot b_0\), \(G_{k_0}\)‘s denote FILR corrections to the Reynolds stress and, for \(b_1 < 1\), we have

\[
G_{k_0} G_{k_0} \delta \phi_2 \delta \phi_2 \simeq \left[ 1 - \left( 3 / 4 \right) (b_{k_0} + b_{k_0}') \right] \delta \phi_2 \delta \phi_2. \tag{28}
\]

\(\delta A_{\perp 1}\), meanwhile, can be readily derived from electron force balance along \(b_0\) and is given by

\[
\delta A_{\perp 2} = (i / 2) \sum_{k' = k_0} \Lambda_{k_0}^{\psi} (\delta A_{k_0}) (\delta A_{k_0}^* / k_0' B_0). \tag{29}
\]

Note, in Eq. (29), we have assumed, having in mind \(k \rho_1 \sim O(1)\), \(k_0^2 \delta e \ll 1\) with \(\delta e = e / \omega_{pe}\) being the electron collisionless skin depth. Equations (27) and (29) may be regarded as the generating equations for, respectively, the ESCC, \(\delta \phi_2\), and MSCC, \(\delta A_{\perp 2}\). Both are, we emphasize, excited by KAW simultaneously.

The dynamics of KAW sidebands is governed by the quasi-neutrality condition and the vorticity equation, which become, for \(b_1 < 1\)

\[
(1 + \tau b_\perp) \delta \phi_{k_\perp} - \delta \psi_{k_\perp} = -i (c / B_0) \Lambda_{k_\perp}^{\psi} (\delta \phi_{k_\perp} / \omega_0) (\delta \phi_{k_\perp} - \delta \psi_{k_\perp}) \tag{30}
\]
\[ k_1^2 \left[ (\omega/c_k) v_A^2 (1 - 3b_0/4) \delta \phi_k - \delta \psi_k \right] = i(c/B_0) (\omega/c_k v_A^2) k_1^2 \delta \phi_{\nu} (\delta \phi_z - \delta \psi_z), \]
\[ \delta \psi_z \equiv (\omega_0 \delta A_{\varepsilon})/c \delta k_0. \]

Equations (30) and (31) can be combined to yield
\[ \epsilon_A \delta \phi_k = i(c/B_0) \Lambda_k^\nu (2b_0/\omega_k (b_0 + b_2)) \delta \phi_{\nu} (\delta \phi_z - \delta \psi_z), \]
with \( \epsilon_A = (\omega/c_k v_A^2 - [1 + (\tau + 3/4)b_2] \) being the KAW dielectric constant for \( b < 1. \)

Noting that \( \Omega_0 \) is a KAW normal mode and letting \( \omega_k = iv_{\nu\varepsilon} \), \( \epsilon_A \) for the KAW sidebands can then be approximated as
\[ \epsilon_A \approx \pm (2\omega_0/k_0^2 v_A^2) (v_{\nu\varepsilon} = \pm \Delta), \]
with \( \Delta \approx (k_0^2 v_A^2/2\omega_0) (\tau + 3/4)b_2 \) being the frequency mismatch between \( \omega_0 \) and \( \Omega_{\varepsilon} \) normal mode frequency. Substituting Eq. (32) along with Eqs. (30) and (33) into Eqs. (27) and (29), we have
\[ \delta \phi_z = -x_\phi (\delta \phi_z - \delta \psi_z)/\left( v_{\nu\varepsilon}^2 + \Delta^2 \right), \]
\[ \delta \psi_z = -x_\phi (\delta \phi_z - \delta \psi_z)/\left( v_{\nu\varepsilon}^2 + \Delta^2 \right), \]
where
\[ x_\phi = \frac{ck k_0^2 \delta \phi_0}{B_0} \left( 2b_0 (\tau + 3/4)(b_0 + b_2) + b_2 \right) \]
\[ \text{and} \]
\[ x_\phi = \frac{ck k_0^2 \delta \phi_0}{B_0} \left( b_0 b_2 (\tau + 3/4) \right) \left( \frac{b_0 + b_2}{b_0 + b_2} \right). \]

Equations (34) and (35) readily yield the following modulational instability dispersion relation:
\[ \gamma_{\nu\varepsilon}^2 + \Delta^2 = -(x_\phi - x_\psi), \]
and, since \( x_\phi > x_\psi, \gamma_{\nu\varepsilon}^2 < 0 \) and it is modulational stable, at least for \( b_2 < 1 \) considered here.

Let us make a few remarks here. First, note that \( |\delta \psi_z|/|\delta \phi_z| \sim O(1) \) and, thus, both ESCC and MSCC are nonlinearly excited by KAW. Now, if one makes the erroneous decoupling assumption, e.g., suppressing \( \delta \phi_z \) artificially, one would obtain the erroneous conclusion that the MSCC is modulational unstable. Meanwhile, if one ignores the FILR corrections to the Reynolds stress, one would, again, obtain the erroneous conclusion that the coupled ESCC-MSCC modulational stability depends on the temperature ratio \( T_i/T_e \). Finally, we remark that, in the case of tokamak plasmas, while \( \delta \phi \) corresponds to the zonal flow, \( \delta \psi \) corresponds to the zonal current. That the present results demonstrates a coupled ESCC-MSCC dynamics, thus, suggests that, in general, both zonal flows and currents must be treated on the same footing, when nonlinear excitations by SAW are considered, as indeed will be shown in Sec. V.

\[
V. \text{ EXCITATION OF ZONAL STRUCTURES BY TOROIDAL ALFVEN EIGENMODES IN TOKAMAK PLASMAS}
\]

In order to examine how geometries break the pure Alfvénic state, we shall consider specifically the case of spontaneous excitation of zonal structures by TAE. Thus, we shall adopt the ideal MHD and incompressibility constraints, and a \( \beta \ll 1 \) tokamak geometry of shifted circular magnetic surfaces. In this model, as noted earlier, SAW is composed of a continuous and discrete spectrum, with TAE being the discrete AE considered here. The zonal structure, meanwhile, is the zero-frequency one. Analysis of the finite-frequency Geodesic Acoustic Mode (GAM) will be left for a future publication.

Within the current model, the vorticity equation for the zonal potential, \( \delta \phi_z \), is given by
\[ -i \omega_x \delta \phi_z = \frac{\omega}{2} \left( \frac{c}{B_0} \right) \sum_{k' + k = k} \Lambda_{k'}^\nu \left( k_{1y}^2 - k_0^2 \right) \delta \phi_{\nu} \delta \phi_z, \]
where \( \omega_x \approx 0.6 \omega \epsilon^{-1/2} b_2 \) is the polarizability enhanced by the trapped ions, \( \epsilon \equiv r/R_0 \) is the ratio between the radial coordinate and the torus major radius. Equation (39) indicates that, while for the SAW continuous spectrum the right-hand side vanishes, as in the pure Alfvénic state, it remains finite for the discrete AEs, such as TAE, i.e., the Alfvénic state could be broken by the AEs. Here, it is also worthwhile noting the close similarity between Eqs. (27) and (39), i.e., the vorticity equation for the zonal (convective cell) potential derived in the case of modulational excitation by KAW, discussed in Sec. IV. This similarity stems from the general structure of kinetic vorticity equation, discussed in Refs. 9, 17, 44-46. In this specific case, the comparison between Eqs. (27) and (39) shows that the novelty introduced by toroidal geometry is the polarizability enhancement by the trapped ions, on the left hand side, and the non-cancellation between Reynolds and Maxwell stresses on the right hand side, due to the TAE frequency shift from the SAW continuous spectrum, caused by the equilibrium variation-induced “defects,” discussed in the Sec. 1. Meanwhile, as toroidal geometry effects are assumed to be the dominant one in this section, the non-ideal kinetic effects, due to FILR corrections borne in the \( G_i \)’s expression [cf. Eq. (28)] as well as in the KAW dispersion relation, are also dropped on the right hand side.

To proceed with the modulational instability analysis, we let, as in Sec. IV, \( \Omega_0, \Omega_\varepsilon, \Omega_{\nu\varepsilon} = \Omega_\nu + \Omega_\varepsilon \), respectively, the pump TAE, the zonal mode, and the TAE sidebands. Following Ref. 47, \( \delta \phi \) can be expressed as
\[ \delta \phi_0 = \alpha \Phi_0 (nq - m) + \text{c.c.,} \]
\[ \delta \phi_z = \alpha e^{i(n \phi_c - m \phi_c)} \sum_m e^{-i m \phi_c} \Phi_0 (nq - m) + \text{c.c.,} \]
\[ \delta \phi_z = \alpha e^{i(n \phi_c - m \phi_c)} \sum_m e^{-i m \phi_c} \Phi_0 (nq - m) + \text{c.c.;} \]
\[ (40) \]
and similarly for the $\delta A_j$. In above equation, the $A^\prime$s denote the radial envelopes, and $\Phi_0$ is the eigenmode function localized about $k_FqR_0 = 1/2$, or half way between two neighboring mode rational surfaces. Equation (39) then becomes

$$io_{\omega_T} z \partial \phi_z = \frac{c}{B_0} k_z k_z r^2 \left( 1 - \frac{1}{4\Omega_0^2} \right) (A_0^\prime A_+ - A_0 A_-),$$  \hfill (41)

and $\delta A_\parallel$ or $\delta \psi_z \equiv (\omega_0 \delta A_\parallel / c k_0)$ of Eq. (29) becomes

$$\delta \psi_z = \frac{c}{B_0} k_z A_0 \left( A_0^\prime A_+ + A_0 A_- \right).$$  \hfill (42)

Note here $\Omega_0 = \omega_0 / \omega_A$, $\Omega_0^2 < \Omega_0^2 < \Omega_e^2$, $\Omega_e^2 = (1/4)(1 \pm \epsilon_0) = 2(1/2R_0 + \Delta S)$, and $\Delta S$ is the Shafranov shift. As to the TAE sidebands, combining the corresponding quasi-neutrality condition and the vorticity equation yields

$$A_0 \pm \epsilon \Delta z = -2i \frac{c}{B_0} k_z \epsilon_0 A_0 \left( \frac{A_0^\prime}{A_0} \right) (\delta \phi - \delta \psi),$$  \hfill (43)

where

$$\epsilon := \frac{\Omega_0^2}{\epsilon_0 \Omega_T^2} \Lambda_T (\Omega) D(\Omega, k_z),$$  \hfill (44)

and

$$\Lambda_T = (\Gamma \Gamma - \Gamma^\prime \Gamma^\prime)^{1/2} = \left[ (\Omega^2 - \Omega_0^2) (\Omega_0^2 - \Omega^2) \right]^{1/2},$$  \hfill (45)

$D(\Omega, k_z) = \Lambda_T (\Omega) - \tilde{\omega}_T (\Omega, k_z)$, and $\tilde{\omega}_T$ corresponds to the equilibrium “defect” in the ideal region, leading to the existence of the TAE bound state.\(^9\)

Following analysis similar to that of convective cells, given in Sec. IV, we can readily derive

$$\partial \phi_z = -\check{\omega}_{\phi} \frac{(\delta \phi - \delta \psi)_z}{\gamma_z^2 + \Lambda_T^2},$$  \hfill (46)

and

$$\delta \psi_z = -\check{\omega}_{\psi} \frac{(\delta \phi - \delta \psi)_z}{\gamma_z^2 + \Lambda_T^2},$$  \hfill (47)

where

$$\check{\omega}_{\phi} = -2 \frac{c}{B_0} k_z k_z |A_0| \left( \frac{1}{4\Omega_0^2} \right) \left( \frac{b_0}{b_+} \right),$$  \hfill (48)

and

$$\check{\omega}_{\psi} = 2 \frac{c}{B_0} k_z k_z |A_0| \left( \frac{b_0}{b_+} \right) \epsilon_0 \Omega_0^4 \Lambda_T (\Omega_0) \partial D / \partial \Omega_0,$$  \hfill (49)

and modulational instability sets in when

$$\gamma_z^2 = \check{\omega}_{\phi}/\check{\omega}_{\psi} - \Lambda_T^2,$$  \hfill (50)

or

$$\left( \frac{c}{B_0} k_z k_z |A_0| \right)^2 \left( \frac{b_0}{b_+} \right) \epsilon_0 \frac{4\Omega_0}{\Lambda_T (\Omega_0) \partial D / \partial \Omega_0} \left( \frac{\Lambda_T}{\Omega_0} \right)^2 \left( \frac{\Lambda_T}{\Omega_0} \right) > 1.$$  \hfill (51)

Note that $\Lambda_T (\Omega_0) \sim (\Omega_0^2 - 1/4) \sim \Omega (\epsilon_0)$ and $|b_0 / \epsilon_0| \sim \Omega (1/4 \Omega_0^2 ) \sim \Omega (1/2 \Omega_0^2)$; we then have $|\delta \phi / \delta \psi| \sim \Omega (1/2 \Omega_0^2)$, i.e., the zonal current generally dominates over the zonal flow due to the trapped ion enhanced polarizability. In the low-$\beta$ regime, typically, $\partial D / \partial \Omega_0 > 0$; the modulational instability condition, Eq. (51), becomes approximately $\Lambda_T / \Omega_0 > 0$ and

$$\left( \frac{c}{B_0} k_z k_z |A_0| \right)^2 \left( \frac{b_0}{b_+} \right) \epsilon_0 \frac{4\Omega_0}{\Lambda_T (\Omega_0) \partial D / \partial \Omega_0} \left( \frac{\Lambda_T}{\Omega_0} \right) > 1.$$  \hfill (52)

for typical tokamak parameters. Equation (53) suggests that the nonlinear excitation of zonal currents could provide a competitive mechanism for saturating nonlinear TAEs.

We remark that from the above analysis it is clear that other discrete AEs could also break the pure Alfvenic state. In addition, we can expect other parametric decay channels, e.g., TAE decays to the GAM, as well as beta-induced Alfvenic eigenmode (BAE) excitations of zonal structures. In this section, we have chosen the case of spontaneous excitation of zonal structures by TAE as example of how geometries break the pure Alfvenic state, since this problem has recently attracted significant attention.\(^41\) However, other examples could be chosen to elucidate the combined actions of the three different effects analyzed so far and discussed in Secs. III–V. The joint effect of plasma compressibility and toroidal geometries, e.g., has been investigated by Hahm and Chen\(^48\) and shown to be responsible of the decay of a pump TAE into a daughter TAE and an ISW in the MHD regime. On the basis of the discussion given in Sec. III, the analysis of Ref. 48 could be readily extended to the kinetic regime and also applied to the case of other AEs.

VI. CONCLUSIONS AND DISCUSSIONS

In the present paper, we propose that examining various effects that could break the pure Alfvenic state provides a useful theoretical framework for considering nonlinear wave-wave interactions among shear Alfven waves. Most of the results reported here, with some exceptions, are published in the literature cited in the text. The most novel element, brought forward by this work, is the formulation of a general theoretical framework based on the properties of the pure Alfvenic state, which can be adopted for understanding the various physics processes underlying the nonlinear dynamics of Alfvenic fluctuations and their manifestations, including their characteristic features expressed by experimental observations.
Specifically, we have examined three effects: finite ion compressibility, non-ideal kinetic effects, and the tokamak geometry, keeping them separate for the sake of clarity. In realistic situations, all these three effects must be considered on the same footing and, depending on the specific problem under investigation, may concur in various extents to the breaking of the pure Alfvénic state and, hence, to the nonlinear system behavior. Some examples of such practical applications are hinted at in the manuscript, although they are still unpublished work and remain subject of ongoing research. In the case of finite ion compressibility, we employed the nonlinear gyrokinetic equations and examined the parametric decay of a pump SAW to a backscattered SAW and an ion sound wave. Our results indicate that, in the kinetic regime, the decay process, due to the FILR effects, is quantitatively enhanced over and qualitatively different from that in the ideal MHD regime, and the results carry significant implications to the wave-induced transports. In the case of non-ideal kinetic effects, we have, again, employed the nonlinear gyrokinetic equation and analyzed the nonlinear excitation of convective cells by KAW. Our results, which, in this case, are novel and unpublished, demonstrate the importance of retaining the coupled electrostatic and magnetostatic convective cells, as well as the FILR corrections to the Reynolds stress. Finally, in the case of tokamak geometry effects, we have analyzed the nonlinear excitation of zonal structures by TAE. Our results, which, in this case, are indeed intellectually challenging and obviously of practical importance in this important area of nonlinear physics of shear Alfvén waves.

ACKNOWLEDGMENTS

This work was supported by US DoE, NSF, ITER-CN, and NSFC grants.