On resonant destabilization of toroidal Alfvén eigenmodes by circulating and trapped energetic ions/alpha particles in tokamaks

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Toroidal Alfvén eigenmodes are shown to be resonantly destabilized by both circulating and trapped energetic ions/alpha particles. In particular, the energetic circulating ions are shown to resonate with the mode not only at the Alfvén speed \(v_A\), but also at one-third of this speed, while for trapped ions, the dominant instability mechanism is shown to be due to the resonance between the precessional magnetic drift and the wave. Implications of the theory for present and future tokamaks are discussed.

With the advent of next-generation fusion experiments which will focus on thermonuclear self-heating, it has become imperative to assess the potential of collective instabilities instigated by alpha particles. There are two classes of instabilities that are believed to be of serious concern to alpha particle confinement: kinetic ballooning modes (KBM)\(^1,2\) and toroidal Alfvén eigenmodes (TAE).\(^3,4\) These modes deserve special scrutiny because they are discrete in character \(\omega_{KBM} \approx \omega_{eB} \) and \(\omega_{TAE} \approx v_A/2qR\), where \(\omega_{eB} \approx k_0 q_0 v_p/L_w, L_w^{-1} = d \ln p/\ln \rho\), is the bulk ion pressure scale length, \(\rho_p = v_p/\Omega\), is the ion Larmor radius, \(v_A = B/(4\pi n_i)\) is the Alfvén velocity, and \(q\) is the safety factor. Hence, unlike modes in the magnetohydrodynamic (MHD) continuum, they experience negligible damping due to phase mixing, and are quite susceptible to instability. In recent work,\(^5\) we have shown that both circulating and trapped energetic ions can resonantly destabilize KBM's. In this Letter, we show that TAE modes can similarly be excited by energetic ion transit (cf. also Refs. 1 and 3) and both precessional drift and bounce resonances for a sufficiently steep hot ion pressure gradient. In particular, we show that energetic circulating ions can resonate with the TAE mode not only at the Alfvén speed, but also at one-third of this speed, while for trapped ions, the dominant instability mechanism is shown to be due to the resonance between the precessional magnetic drift and the wave. Implications of the theory for present and future tokamaks are discussed.

We now consider how the energetic particles resonate with the MHD wave. First focus on circulating particles. Substituting the eigenmode expression for \(S\) into the gyrokinetic equation and solving for the distribution function, we obtain

\[
\delta\psi = \exp(-\Lambda|\partial_1|)[\Delta \cos(\partial_0/2) + 2 \sin(\partial_0/2)],
\]

where \(\Lambda = (-\Gamma_- \Gamma_+)^{1/2}, \Gamma_\pm = \Omega^2(1 \pm \epsilon) - 1/4 \sim O(\epsilon),\) and \(\Delta = 4/\pi\hat{s}^2\).

We now consider how the energetic particles resonantly destabilize the MHD wave. First focus on circulating particles. Substituting the eigenmode expression for \(\delta\psi\) into the gyrokinetic equation and solving for the distribution function, we obtain

\[
\frac{d^2}{d\theta^2} + \Omega^2(1 + 2 \epsilon \cos \theta) + \frac{\Delta}{f^2(\theta)} - \left(\hat{s} - \Delta \cos \theta\right)^2 \frac{f^2(\theta)}{f^2(\theta)} - \frac{4\pi e_\psi \omega}{c} \int d\theta \omega_{AB} \delta g_h = 0,
\]
\[
\delta_{h,u} = \frac{i \epsilon_h}{2m_h} \frac{Q \hat{d}_{dh}}{\omega_{\omega_t}} f_0^{-1/2}(\theta_1) 
\times \sum_{l=1,2,3/2} \exp \left( -\frac{\Lambda |\delta_0|}{(\omega/\omega_t)^2 - l^2} \right) \left( \frac{\omega}{\omega_i} C_{i+1/S_i} \cos \delta_{\theta_0} \right) \cos l \theta_0 
+ \left( \frac{\omega}{\omega_i} S_{i-1} \right) \sin l \theta_0 ,
\]

where

\[
C_{1/2} = \Delta' + 2\delta_{0}, \quad C_{3/2} = \Delta' - 2\delta_{0}, \quad S_{1/2} = -2 + \Delta' \delta_{0}, \quad S_{3/2} = 2 - 2\delta_{0} ;
\]

\[
Q_{h} = (\omega_{\phi} + \omega_{i}) F_{0h} \quad \omega_{i}, -\omega_{i}/q_R \quad \text{(the transit frequency, } f_0 = 1 + 3 \delta_{0} \text{, and the subscript (u) tr denotes (un) trapped ions.})
\]

It is immediately clear from Eq. (2) that two resonances are possible: \( l \theta_0 = \pm \omega_{\phi} / \omega_{i} \), and \( \omega_{\phi} / \omega_{i} \). Physically, these resonances can be understood as follows. The TAE mode frequency, given by \( \omega_{\phi} = \omega_{A} / 2q_R \), is completely determined by bulk plasma dynamics as discussed above. The destabilizing “force” associated with the energetic particles, on the other hand, is given by \( \omega_{\phi} = \omega_{A} / 2q_R \), which generates oscillations at \( \mp \omega_{\phi} / \omega_{i} \). Parametric pumping occurs when the effective transit frequency of the energetic ions, thus given by \( \omega_{\phi} / \omega_{i} \) and \( \omega_{\phi} / \omega_{i} \), matches the frequency of the MHD waves; hence, the resonances at the fundamental and one-third Alfvén speed. Equation (2) is now substituted into Eq. (1), and the eigenmode equation is solved by asymptotically matching the ideal region solution to the gap region solution. The resulting dispersion relation, to leading order, is

\[
\Gamma = \frac{\pi q^2 R^2 \epsilon_h^2}{2} \frac{m_h}{\epsilon_h} \sum_{m=\pm} \int d\theta \quad \cos l \theta_0 \frac{Q \hat{d}_{dh}}{\omega_{\omega_t}} \left[ \delta (\theta_{\omega} - \sigma \theta_{A}) + \frac{1}{3} \delta (\theta_{\omega} - \frac{\theta_{A}}{3}) \right].
\]

Adopting an isotropic, hot ion slowing-down distribution function, \( F_{0h} = (B \beta_{0} / n_{0}^{1/2}) \) [bars represent the maximum (birth) energy and \( E_{cr} = E_{B} / E_{n} \), \( E_{n} \) is the energy at which the ions change from slowing down on the electrons to pitch angle scattering off the bulk ions], we obtain the following expression for the growth rate:

\[
\gamma^{(\omega)} \frac{m_h}{\omega_A} = - \frac{3}{2} \frac{3 \nu_A L_{ph}}{2q_k \omega_h \nu_m R},
\]

where frequencies decorated with a subscript \( m \) denote their value at \( \nu_m = v_m^2 / 2 \) and it is assumed that \( \varepsilon = 2e\nu_{h} / \nu_{m} \leq 1 \) (for \( \varepsilon > 1 \), the growth rate is reduced by a factor \( \varepsilon^{-1/2} \)). For \( \nu_{h} > \nu_{m} > \nu_{A} / 3 \), the growth is reduced from the value given in Eq. (4) by a factor \( \gamma^{(\omega)} \), and if \( \nu_{A} / 3 > \nu_{m} \), the drive becomes even more feeble (higher order in \( \varepsilon \)). The most unstable wavelength corresponds to \( k_{0} \rho_{ph} < (\varepsilon / \delta_{h}) \nu_{m} / \nu_{h} \), yielding

\[
\delta_{\theta_0} = \frac{\pi q_{h}}{2} \frac{\gamma^{(\omega)}}{\omega_{A}} n_{A} \nu_{A} R_{ph},
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\delta_{\theta_0} = \frac{\pi q_{h}}{2} \frac{\gamma^{(\omega)}}{\omega_{A}} n_{A} \nu_{A} R_{ph},
\]


\( \omega_r = \bar{\omega}_{dr} \), the resonant energy is given by

\( E_{res}/E_m = (1/2 q k_0 \rho_h) v_A/v_m \). Clearly, there is a maximum

wavelength (corresponding to the birth energy) above which the resonance condition cannot be satisfied, i.e.,

\( (k_0 \rho_h)_{\text{min}} = v_A/2 q v_m \). After some manipulation, the growth rate can be derived:

\[
\gamma_{(P)}(\omega_A) = q^2 I_0 \Gamma_{\text{se}} \beta_h_{nr} \overline{p} \left( \frac{k_0 \rho_h}{2 q v_m} J_0 \left( \frac{k_0 \rho_h v_A}{2 \epsilon v_m} \right) \right) x \left( \frac{v_A}{v_m} \right) \left( 1 - \frac{L_{ph}}{R} \right),
\]

(8)

where \( I_0 \) is a pitch angle integral of order unity. The instability sets in at \((k_0 \rho_h)_{\text{min}}\) and decays at shorter wavelengths due to FLR and FBW averaging. The maximum growth rate, corresponding to the longest wavelength, is given by

\[
\gamma_{(P, \text{max})}(\omega_A) = 2q I_0 \Gamma_{\text{se}} \beta_h_{nr} \overline{p} R \left( \frac{1}{2 \epsilon v_m} \right) J_0 \left( \frac{v_A}{v_m} \right) \left( 1 - \frac{L_{ph}}{R} \right).
\]

(9)

Note that the instability becomes weaker with increasing \( v_A/v_m \) and pitch angle. For the bounce resonance (\( p = 0 \)), on the other hand, the instability is independent of wave-length: \( E_{res}/E_m = v_m^2/(4 p v_A^2 \epsilon) \). The dominant contribution for this type of resonance comes from \( p = \pm 1 \), and requires \( v_m > v_A/2 \epsilon^{1/2} \). This is because (a) higher bounce harmonics bring lower-energy particles into resonance, for which the instability is mitigated as discussed in Ref. 2, and (b) higher bounce harmonics cause more rapid phase mixing of the bounce-average integrand. The growth rate is then

\[
\gamma_{(B)}(\omega_A) = \frac{2 \epsilon q^2}{9 \cdot 2 \Gamma_{\text{se}} \beta_h_{nr} \overline{p}} \left( \frac{k_0 \rho_h}{\epsilon v_m} \right) \left( \frac{v_A}{v_m} \right) \overline{p} \left( \frac{L_{ph}}{R} \right) x \left( \frac{L_{ph}}{R} \right) \overline{p} \left( \frac{L_{ph}}{R} \right)
\]

(10)

In this case, the instability is maximized for \((k_0 \rho_h)_{\text{max}} = (2 \epsilon / q) (v_A/v_m)\). The maximum growth rate is then given by

\[
\gamma_{(B, \text{max})}(\omega_A) = \frac{2 \epsilon q^2}{9 \cdot 2 \Gamma_{\text{se}} \beta_h_{nr} \overline{p}} \left( \frac{k_0 \rho_h}{\epsilon v_m} \right) \left( \frac{v_A}{v_m} \right) \overline{p} \left( \frac{L_{ph}}{R} \right) x \left( \frac{4 \epsilon v_m}{v_A} \right) \overline{p} \left( \frac{L_{ph}}{R} \right).
\]

(11)

Comparing Eq. (9) to Eq. (11), we see that the drift resonance instability is an order of magnitude larger than the bounce resonance instability, predominantly due to enhanced phase mixing of the bounce integrand.

It is useful to compare the energetic ion destabilized TAE instability with KBM's.\(^5\) The crucial difference is that, in the region of energetic particle concentration, the latter are triggered only when the bulk plasma is locally close to the first stability boundary for ideal ballooning modes, whereas TAE destabilization is expected only below this boundary. Thus, our analysis predicts that the two modes do not coexist at the same spatial locus and that in practice, there will be a transition from one mode to the other with increasing beta or input power. The critical \( \beta_h \) for TAE instability is determined from a balance between the growth rates calculated here, and the sum magnitude of continuum damping,\(^6\) trapped electron collisional damping,\(^6\) and ion Landau damping. On the basis of the calculations presented here, we can conclude that the critical \( \beta_h \) for instability will exhibit an absolute minimum in the vicinity of the fundamental resonance, i.e., \( v_m = v_A \). For birth speeds,\(^7\) this value, \( \beta_{h,cr} \), increases because now only the one-third speed resonance for transit ions or higher bounce harmonics for trapped ions can satisfy the resonance condition (there will be local minima at these resonances, but they may be smeared out because of finite mode widths), and FBW/FLR averaging mitigates the precessional drift resonance drive. For birth speeds,\(^7\) this value, the threshold also goes up because now progressively less energetic ions can be tapped by the resonance.

Finally, the theory developed in this Letter predicts that contrary to conventional wisdom, trapped energetic particles can destabilize TAE modes. In this context, it should be noted that the precessional drift resonance drive for the case of high-power \(^3\)He minority ICRF heating experiments on the Joint European Torus (JET),\(^10\) which generated 1 MeV tail ions \( [v_{\text{th}} \approx (1 \text{ MeV})/v_A \approx 0.8] \), is mitigated by strong FBW averaging effects \( [\text{cf. Eq. (9)}] \). For JET D–T parameters, on the other hand, \( v_{\text{th}}/v_A = 2 \), making the instability more viable. It follows that previous ICRF experiments on JET were not optimal for TAE excitation and hence, it would be short-sighted to dismiss the likelihood of collective alpha-particle instabilities in the D–T phase on the basis of the existing evidence. By changing certain plasma parameters (e.g., reducing the field, increasing the background density, and/or reducing the \(^3\)He or H minority concentration, thus boosting the tail ion energy), on the other hand, it may be possible to trigger the trapped ion TAE resonance instability on JET with ICRF.

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7The assumptions that energetic particles be treated perturbatively and weak shear are not always satisfied experimentally. Indeed, they are not critical for the conclusions of the present work, and merely serve to simplify the analysis.
10J. Jacquinot and G. Sadler (private communication).