Linear and nonlinear simulations of electron fishbone in tokamak equilibria using XHMGC

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Outline

• Introduction
  – motivations, relevance

• XHMGC: eXtended Hybrid MHD Gyrokinetic Code
  – what’s new, energetic electron distribution function, …

• Linear Simulations
  – energetic electron density profile peaked on-/off-axis
  – identification of the characteristic resonances

• Nonlinear Simulations
  – investigation of the nonlinear dynamics of the mode
    through Hamiltonian mapping techniques
  – identification of saturation mechanism
Introduction

• Internal kink instabilities exhibiting fishbone like behaviours have been observed in a variety of experiments, where a high-energy electron population was present, due to, e.g., ECRH and/or LHH/LHCD

• These e-fishbones have linear dispersion relation and excitation mechanisms that are similar to those of energetic ion driven fishbones

• They are characterized by a very small ratio between the resonant particle orbit width and the characteristic fishbone length scale (~ rigid radial kink-type displacement)

• This feature is also expected to characterize ion fishbones in burning plasmas of fusion interest due to the large plasma current, while it is not realized for the energetic ions in present-day experiments

• Moreover, fluctuation induced transport of magnetically trapped resonant particles, due to precession resonance, is expected to depend on energy and not mass because of the bounce averaged dynamic response

• These analogies between e-fishbones in present-day devices and fishbones in burning plasmas make the investigation of the former a good basis to study and predict the effects of the latter.
XHMGCG model

• Original HMGCG model:
  – Thermal plasma:
    - $O(\varepsilon^3)$ nonlinear reduced MHD equations,
    - circular shifted magnetic surface equilibrium,
    - zero bulk plasma pressure;
  – Energetic particles described by nonlinear guiding-center Vlasov equation (particle-in-cell technique).

• HMGCG has been recently eXtended (XHMGCG) to include new physics:
  – thermal ion compressibility (PIC) and diamagnetic effects (in order to allow for an entirely novel treatment of the realistic (enhanced) inertia response and ion Landau damping)

• XHMGCG is now able to treat simultaneously three independent particle populations kinetically (PIC), assuming different equilibrium distribution functions (as, e.g., bulk ions and electrons, energetic ions and/or electrons accelerated by NB, IRCH, ECRH, alpha’s, etc.).
XHMGC model

• Addressing electron fishbone excitation poses the numerical challenge of properly handling the effective bounce averaging of linear and nonlinear mode dynamics, due to the extremely fast parallel electron motion along equilibrium magnetic field lines, which has been addressed and solved by sub-cycling in the particle-in-cell numerical simulation scheme.

• XHMGC allows to follow, in a self-consistent simulation, a set of test particles; the phase-space coordinates of such particles are stored in time, and can be used to compute a variety of single particle physical quantities as, e.g., the single particle frequencies of the supra-thermal electrons, namely, the precession and bounce frequencies. The resonances underlying the linear instability will be clearly identified.

• The use of energetic particle phase-space diagnostics, based on the Hamiltonian mapping techniques generating kinetic Poincaré plots, will allow us to clearly identify the physics process underlying fishbone mode saturation, frequency chirping and secular (versus diffusive) energetic particle redistribution.
Energetic particles

- **Energetic electrons**, treated kinetically (PIC module):

\[
f_{\text{electrons}} \propto \frac{n_{\text{Ee}}(\psi)}{T_{\text{Ee}}(\psi)^{3/2}} \Xi(\alpha; \alpha_0, \Delta) e^{-E/T_{\text{Ee}}(\psi)} \equiv \frac{n_{\text{Ee}}(\psi)}{T_{\text{Ee}}(\psi)^{3/2}} \hat{f}_{\text{electrons}}
\]

\[
\Xi(\alpha; \alpha_0, \Delta) \equiv \frac{4}{\Delta \sqrt{\pi}} \exp \left[ - \left( \frac{\cos \alpha - \cos \alpha_0}{\Delta} \right)^2 \right] \frac{\Delta}{\sqrt{\pi}} \text{erf} \left( \frac{1 - \cos \alpha_0}{\Delta} \right) + \text{erf} \left( \frac{1 + \cos \alpha_0}{\Delta} \right)
\]

anisotropy \( \Xi \): \( 1/\Delta \) determines the width of \( f_{\text{electrons}} \) along the \( \alpha_0 \) direction in the \( (v_{\parallel}, v_{\perp}) \) plane

\[
E = \frac{1}{2} m_e v_{\parallel}^2 + \mu \omega_{ce} ; \quad \cos \alpha \equiv \frac{v_{\parallel}}{\sqrt{2E/m_e}} ; \quad \sin^2 \alpha \equiv \frac{\mu \omega_{ce}}{E}
\]

\[
\hat{u} \equiv v_{\parallel}/v_{\text{th},e,0} \quad \hat{\mu} \equiv \mu \omega_{ce0}/T_{\text{Ee}0} ; \quad \mu = v_{\perp}^2/(2B)
\]

\[
v_{\text{th},e,0} = \sqrt{T_{\text{Ee}0}/m_e} \quad \omega_{ce} = eB/(m_ec)
\]

\[
T_{\text{Ee}} = T_{\text{Ee}0} = 50 \text{ keV} \quad \rho_{\text{Ee}}/a \approx 3.52 \times 10^{-4} \quad v_{\text{th},e,0}/v_{A0} \approx 11.27 \quad v_{A0} = B_0/\sqrt{\mu_0 m_i n_0}
\]
Energetic particles

• Two different energetic electrons density profiles:

  - Peaked on-axis: $n_{Ee}(r)$
  - Peaked off-axis: $n_{Ee}(r)$

• Strongly anisotropic energetic electrons distribution functions $f_{electrons}$:

  (1) Mostly deeply trapped
      $\cos \alpha_0 = 0, \Delta = 10$

  (2) Mostly trapped
      $\cos \alpha_0 = 0, \Delta = 2.5$

  (...) Mostly counter-passing
      $\cos \alpha_0 = -0.6, \Delta = 10$

  (...) Mostly co-passing
      $\cos \alpha_0 = +0.6, \Delta = 10$
• What can we expect from linear theory?

– stability of the fishbone mode given by $\delta W_k$:

\[
\delta \hat{W}_k \propto I \cdot \frac{1}{n \overline{\omega}_d + \ell \omega_b - \omega} Q F_0 \quad \text{trapped particles}
\]

\[
\delta \hat{W}_k \propto I \cdot \frac{1}{n \overline{\omega}_d + [\ell + n q(r) - m] \omega_t - \omega} Q F_0 \quad \text{circulating particles}
\]

\[
Q F_0 = \omega \left( \partial_E + \frac{\hat{\omega}^* E_s}{\omega} \right) F_0, \quad \hat{\omega}^* E_s F_0 = \frac{1}{\omega_c} \frac{k \times B}{B} \cdot \nabla F_0
\]

• condition for instability: $\hat{\omega}^* E_s / \omega > 0$

• real frequency given by the resonant denominator of $\delta W_k$

$I \circ$: integro-differential operator

$\omega_b$ the bounce frequency of trapped particles

$\omega_t = v_{||}/q R_0$ the transit frequency of circulating particles

$\overline{\omega}_d$ is bounce averaged toroidal precession frequency

$\omega^* E_s \approx \frac{m_{pol}}{n e_s r B} \frac{dp}{dr}$ the diamagnetic drift frequency; note that sign depends on the electric charge of the $s$ species ("$i$", "$e$") and on the radial gradient of pressure (density)
• **Bulk ions:**
  - \(n_{i0} = 1 \times 10^{20} \text{ m}^{-3}\),
  - \(n_i(\psi)/n_{i0} = (1-\psi)^{1/2}\),
  - \(T_{i0} = 2 \text{ keV}\),
  - \(T_i(\psi)/T_{i0} = (1-\psi)\),
  - isotropic maxwellian (when treated kinetically)

• **Equilibrium:**
  - \(\varepsilon \equiv a/R_0 \approx 0.3\);
  - \(q_0 \approx 1.25, q_{\text{min}} \approx 1.05, r_{q-\text{min}}/a \approx 0.35, q_a \approx 6\);
  - \(B_T = 5.4 \text{ T}\)
  - simulation will keep \(n=1, m=1, 2, 3, 4, 5, 6\)
  - (use FTU with LH heating as reference for inverted \(q\) profile)
Linear dynamics 1

- On-axis energetic electrons density profile:

- Deeply trapped energetic electrons ($\cos \alpha_0=0$, $1/\Delta=10$):

- Mode propagates in the electron diamagnetic direction:

- Power exchange particles/wave in the normalized ($\mu, u$) space (at $\theta=0$) of energetic electrons: deeply trapped energetic electrons drive the mode (red region)

Power exchange

Energetic electrons

Trapped/circulating boundaries:
- $r/a=0.14$ (solid line);
- $r/a=0.19$ (dashed line)
Check the characteristic frequencies of energetic electrons using test particles during self-consistent simulation: a set of test particles, initialized such to cover uniformly the \((\mu,u)\) space, are followed in time

- The bounce frequencies \(\omega_b\) of the test energetic electrons are always much larger, in absolute value, than the mode frequency, whereas,
- the precession frequency \(\omega_D\) of the trapped test electrons falls in the range of the mode frequency;
- superimposing the curves \(\omega_D(\mu,u) - \omega = 0\) to power exchange pattern of trapped energetic electrons: very good agreement when mode frequency is considered (\(\omega = -0.095 \omega_{A0}\), black, solid line);
- also plotted the curves \(\omega_D(\mu,u) - (\omega \pm 3\gamma) = 0\), in order to give a qualitative feeling of the natural spread in real frequency induced by the finite mode growth rate (\(\gamma/\omega_{A0} \approx 0.02\) for the case shown). Note that fishbones could be considered one special case of EPM (Energetic Particle driven Mode);
- the regions in\((\mu,u)\) space where the maximum power exchange occurs (red colour in the figure), compare well with the curve which corresponds to the mode frequency observed in the simulation (black, thick curve).
Linear dynamics 2

- Off-axis energetic electrons density profile:

![Graph of \( n_{Ee}(r) \) vs. \( r/a \)]

- In the central region of the plasma column \((0 < r/a \lesssim 0.5)\) the density gradient is positive: \( \dot{\omega}_{Ee} \) change sign with respect to the on-axis energetic electrons density profile case

- Hence, we expect instability for modes having opposite real frequency:
  - mode propagation should be in the ion diamagnetic direction
  - from theory, it is known that trapped particles experience inversion of the toroidal precession frequency if they lie close to the trapped/circulating boundary (barely trapped/circulating particles)

- Trapped energetic electrons \((\cos \alpha_0 = 0, \ 1/\Delta = 2.5):\)
  - reduce \(1/\Delta\) in order to increase the number of particles close to the trapped/circulating boundary
Linear dynamics 2

- Mode propagates in the ion diamagnetic direction (weak rotation):
- Power exchange particles/wave in the normalized \((\mu, u)\) space (at \(\theta = 0\)) of energetic electrons: barely trapped/circulating energetic electrons drive the mode (red region).
- dominant \(m=2\) component, eigenfunction very “spiky”

\[ \phi_{m,n}(r) \]

\[ \varphi(r,\theta) \]

\[ |\varphi(\omega, r)|^2 \]

\[ \hat{u} \]

- “Artificially” switching off all the Fourier components of the energetic electrons contributions but \(m=1\): a subdominant growing mode emerges.
- both modes are very weakly growing, dependence on MHD equilibrium details, mode-mode coupling, …

Power exchange

Trapped/circulating boundary: \(r/a = 0.313\)
Let’s now consider the extended vorticity equation, and highlight the contribution of the different terms included:

\[
\rho_b \left[ \frac{\partial}{\partial t} + \left( \frac{b \times \nabla P_{0i}}{\rho_b \omega_{ci}} + \delta v_b \right) \cdot \nabla \right] \delta v_b = - (\nabla \cdot P_e) - (\nabla \cdot P_i) - (\nabla \cdot P_{Ee}) + \left( \frac{J \times B}{c} \right)
\]

- **diamagnetic, bulk ions**
- **diamagnetic frequency of bulk ions (small)**
- **contribution from bulk electrons (PIC)**
- **drive of the mode (PIC)**
- **Landau damping and enhanced inertia of bulk ions (PIC)**
• Extended vorticity equation:

\[
\rho_b \left[ \frac{\partial}{\partial t} + \left( b \times \nabla P_{0i\perp} + \delta v_b \right) \cdot \nabla \right] \delta v_b = - \left( \nabla \cdot P_i \right)_{\perp} - \left( \nabla \cdot P_{Ee} \right)_{\perp} + \left( \frac{J \times B}{c} \right)_{\perp}
\]

- Extended vorticity equation:

- Only Energetic electrons, peaked on-axis, \( n_{Ee0}/n_{i0} = 0.13 \)
- Mode propagates in the electron diamagnetic direction
- Power exchange particles/wave in the normalized \((\mu, u)\) space (at \(\theta = 0\)) of supra-thermal electrons (0.14 \(\leq r/a \leq 0.19\))
- deeply trapped hot electrons drive the mode (red region)
- solid line: trapped/circulating boundary for \(r/a \approx 0.14\)
- dashed line: trapped/circulating boundary for \(r/a \approx 0.19\)
- Extended vorticity equation:

\[ \rho_b \left[ \frac{\partial}{\partial t} + \left( \frac{\mathbf{b} \times \nabla P_{0i} \perp}{\rho_b \omega_{ci}} + \delta \mathbf{v}_b \right) \cdot \nabla \right] \delta \mathbf{v}_b = - (\nabla \cdot \mathbf{P}_e \perp) - (\nabla \cdot \mathbf{P}_i \perp) - (\nabla \cdot \mathbf{P}_{Ee} \perp) + \left( \frac{\mathbf{J} \times \mathbf{B}}{c} \right) \perp \]

\[ \text{diamagnetic, bulk ions} \]

- Energetic electrons + bulk ions, \( n_{Ee0}/n_{i0} = 0.15 \)
- kinetic contribution of bulk (thermal) ions:
  - Landau damping
  - enhanced inertia
- Power exchange particles/wave: passing bulk ions provide damping

- Energetic electrons
- bulk electrons
- Energetic electrons

\[ \hat{\varphi}_{m,n}(r) \]

\[ \varphi(r, \theta) \]

\[ \omega/\omega_{A0} \]

\[ \gamma/\omega_{A0} \]

\[ \gamma/\omega_{A0} \text{ En. el.} \]

\[ \gamma/\omega_{A0} \text{ En. el.} + \text{th. ions (} T_{i0}=2\text{keV}) \]

\[ n_{He0}/n_{i0} \]

\[ n_{He0}/n_{i0} \text{ En. el.} \]

\[ n_{He0}/n_{i0} \text{ +th. ions (} T_{i0}=2\text{keV}) \]

- \( \varphi(x,y,0) \text{ and } \psi(y,x,0) \)

- HMGG code - ENEA - Frascati

- \( |\varphi(\omega,r)|^2 \)

- \( \omega/\omega_{A0} \)

- HMGG code - ENEA - Frascati

- \( \gamma/\omega_{A0} \text{ En. el.} \)

- HMGG code - ENEA - Frascati

- HMGG code - ENEA - Frascati

- \( \mathbf{U} \)

- \( \hat{\mathbf{U}} \)

- \( \hat{\mathbf{U}} \)
• Extended vorticity equation:

\[
\rho_b \left[ \frac{\partial}{\partial t} + \left( \frac{\mathbf{b} \times \nabla P_{0i\perp}}{\rho_b \omega_{ci}} + \delta \mathbf{v}_b \right) \right] \cdot \nabla \delta \mathbf{v}_b = -\left( \nabla \cdot \mathbf{P}_e \right)_{\perp} - \left( \nabla \cdot \mathbf{P}_i \right)_{\perp} - \left( \nabla \cdot \mathbf{P}_{Ee} \right)_{\perp} + \left( \frac{\mathbf{J} \times \mathbf{B}}{c} \right)_{\perp}
\]

diamagnetic, bulk ions

• Energetic electrons + Bulk ions + Bulk electrons, \( n_{Ee0}/n_{i0} = 0.13 \)

• Power exchange particles/wave: passing bulk electrons contribute to the drive
Linear dynamics 3

- Power exchange between particles and wave:

  - Energetic electrons: drive from deeply trapped particles; some damping from barely trapped/circulating particles

  - Bulk ions: damping from co- and counter-passing particles

  - Bulk electrons: co- and counter-passing bulk electrons contribute to both damping and drive

\[
0.14 \leq r/a \leq 0.19 \quad 0.23 \leq r/a \leq 0.28 \quad 0.37 \leq r/a \leq 0.42 \quad 0.47 \leq r/a \leq 0.52
\]
Nonlinear dynamics

- Let’s now consider the nonlinear phase of the simulation
  - we refer to the on-axis peaked energetic electrons radial density profile
  - deeply trapped distribution function ($\cos \alpha_0=0$, $1/\Delta=10$.)

- We want to identify the physics processes underlying nonlinear mode dynamics and energetic electron transports well above the fishbone excitation threshold, thus we neglect MHD nonlinearity but fully account for nonlinear wave-particle interactions nonpertubatively
Nonlinear dynamics

- (Absolute) frequency chirps down during saturation
- Subsequent bursts appear at larger $|\omega|$ after the first saturation (note that sources are not included in these simulations)
Nonlinear dynamics

- Energetic electrons transport:
  - the radial density profile widens during nonlinear saturation

- velocity space diffusion is also significant during nonlinear saturation for both circulating and trapped particles

$W_{tot\,m,n}$

$n_{Ee}(r)$

$n_{Ee}(r,\theta)$, circulating

$n_{Ee}(r,\theta)$, trapped

$t\omega_A=550$

$t\omega_A=900$

$t\omega_A=1248$
Phase-space dynamics

- Following White [Commun. Nonlinear Sci. Numer. Simulat., 17, (2012) 2200–2214.], we select a set of test particles characterized by certain values of the two constants of the (perturbed) motion: \( \mu \) and \( C = \omega P_\phi - nE \), where \( P_\phi \) is the toroidal canonical angular momentum, \( E \) is the particle energy and \( n \) is the toroidal mode number (this form of the \( C \) constant is the lowest order expression for the conserved Hamiltonian in the extended phase space).

- The chosen \( \mu \) and \( C \) correspond to strong power transfer between energetic electrons and wave during linear phase \((r/a \approx 0.18, \mu=3, \nu=-0.3)\).

- Different test particles are then loaded with different radial coordinates within a shell around the mode peak and corresponding different values of \( \nu_{||} \).

- Plot kinetic Poincaré plots by drawing a dot each time \( t = \hat{t} \) a test particle completes a full banana or transit orbit in the poloidal plane by crossing the outer (\( \theta=0 \)) equatorial plane.

- Plots are given in \((\Theta, P_\phi)\) plane (wave-particle phase \( \Theta \), and toroidal canonical momentum \( P_\phi \)):

\[
\Theta(\hat{t}) = \int_0^{\hat{t}} \omega(t) dt - n\phi(\hat{t}) \quad P_\phi = m_e R\nu_{||} - \frac{e}{c}\psi
\]

- \( \phi \) is the toroidal angle of the test particle.
Some consideration on the kinetic Poincaré plots:
- \(-0.13 \leq P_\phi \leq 50\) roughly correspond to \(0.1 \leq r/a \leq 0.3\)
- Test particles are colored corresponding to their birth value of \(P_\phi\)
- \(P_\phi\) is an invariant of the unperturbed motion, only
- Test particles close to resonance tend to move on “trapped” trajectories
- Test particles far from resonance fast drift in \(\Theta\)
- The separatrix structure drifts significantly upward in the \((\Theta, P_\phi)\) plane (i.e., outward in radius) while proceeding in the nonlinear phase
Phase-space dynamics

• Examples of resonant test particle trajectories:

  • Test particles trajectories in the \((\Theta, P_\phi)\) plane: colour of dots correspond to the instantaneous wave-particle power exchange (red: drive; violet: damping)
  • As the effect of the perturbed field becomes relevant, \(P_\phi\) is no longer conserved
  • Bounce time \(\tau_B\) of the same order of saturation time \(\tau_{NL}\)
  • Power transfer from particles to wave does not average to zero and is connected to rapid frequency chirping
  • Saturation can not be ascribed to “fast” averaging of particles trapped in the wave
Phase-space dynamics

- Evolution of the precession frequency of the linearly resonant particles:
  - consider the average of the precession frequency \( \omega_{D_i}(t) \) of all the \( i \) linearly driving resonant particles weighted with the average power transfer during linear phase \( p_i^{\text{lin}} \) (solid red curve):
    \[
    \bar{\omega}_D(t) \equiv \frac{\sum_i p_i^{\text{lin}} \omega_{D_i}(t)}{\sum_i p_i^{\text{lin}}},
    \]
  - and spread (dashed red curves):
    \[
    \bar{\omega}_D \pm \delta \omega_D; \quad \delta \omega_D(t) \equiv \left\{ \sum_i p_i^{\text{lin}} [\omega_{D_i}(t) - \bar{\omega}_D(t)]^2 / \sum_i p_i^{\text{lin}} \right\}^{1/2},
    \]
  - compare with the instantaneous mode frequency \( \omega(t) \) as measured from simulation (solid black curve, left frame): mode frequency \( \omega(t) \) closely follows \( \bar{\omega}_D(t) \) in order to minimize resonance detuning, i.e. \( |\dot{\Theta}| \), and maintain the “phase-locking” condition, consistently with the mode particle pumping mechanism.
  - Average \( |\dot{\Theta}| \) and its spread computed with actual \( \omega(t) \) (black curves), compared with \( |\dot{\Theta}| \) as obtained by using constant \( \omega \) from linear phase \( \omega_{\text{lin}} \) (red curve): in this case particles would detune much faster w.r.t. the mode\((|\dot{\Theta}| \neq 0)\).
  - Note that phase-locking is the reason for frequency chirping (phase-locking is not exact, see resonance broadening).
Phase-space dynamics

In spite of phase-locking, the mode eventually cannot preserve the initial drive, because the radial displacement of the linearly resonant particles causes them to experience a reduced mode amplitude, due to radial mode structure and radial nonuniformities.

This can be seen from figure, comparing the radial average position of the linearly driving resonant particles $r(t)$ and the spread $r(t) \pm \delta r(t)$ with the linear-phase radial structure of the poloidal electric field $E_\theta$ ($m=1$ component filled in red).
Conclusions

- The XHMGC code have been applied to the study of energetic electron driven fishbones
- **Linear dynamics** - Several features have been studied:
  - anisotropy of energetic electron distribution function;
  - radial density profiles (on- and off-axis peaking);
  - kinetic contribution of energetic electrons (drive), bulk ions (Landau damping and enhanced inertia), bulk electrons (drive/damping);
  - detailed comparison with single particle resonances using test particles diagnostics.
- The results of the first **nonlinear numerical simulation** of energetic electron driven internal kink instabilities have been presented:
  - the mode is excited by precession resonance with magnetically trapped electrons;
  - counter-passing thermal ions tend, instead, to stabilize the mode via Landau damping;
  - the nonlinear dynamics of the mode has been investigated through Hamiltonian mapping techniques;
  - saturation can not be ascribed to “fast” averaging of particles trapped in the wave;
  - nonlinear fishbone dynamics is governed by frequency chirping due to phase-locking, accompanied by mode particle pumping, originally introduced by White et al., (1983) Phys. Fluids 26 2958;
  - the ejection of resonant particles from the region where the wave–particle power exchange is initially localized, produces radial decoupling of resonant particles from the mode structure.
Extra slides
• Estimate the relative importance of “fluid” vs. “kinetic” nonlinearities

• consider the general fishbone like dispersion relation (GFLDR): 
  \[ i |s| \Lambda_n = \delta W_{nf} + \delta W_{nk} \]
  with \( \Lambda_n(\omega) \) the generalized inertia term, \( n \) is the toroidal mode number, and \( \delta W_{nf} \) and \( \delta W_{kk} \) the “fluid” and “kinetic” contribution to the potential energy \( \delta W \)

• compare \( \Lambda_{NL}^n \), i.e. the contribution of MHD wave-wave couplings to the generalized inertia in the singular layer region from Ödbloem et al., Phys. Plasmas (2002) (which ignores kinetic thermal ion and geometry effects), with the nonlinear contribution \( \delta W_{kf} \) of supra-thermal particles to the potential energy in the regular region from Zonca et al., Nuclear Fusion (2007):

\[
\Lambda_{NL}^n \sim \frac{|\delta \xi_{r0}|^2}{\Delta^2} \Lambda \\
\sim \frac{|\delta \xi_{r0}|^2}{r_s^2(\gamma_L/\omega_0)^2} \frac{s^2}{\Lambda}
\]

having made use of the expression for the inertial layer width:

\[
\Delta \sim r_s \left( \frac{\Lambda}{s} \right) \frac{\gamma_L}{\omega_0}
\]

\[
\delta W_{nk}^{NL} \sim \Im \delta W_{nk}^L \frac{|\delta \xi_{r0}|^2}{r_s^2(\gamma_L/\omega_0)^2} \sim \frac{R_0}{r_s} \beta_{Er} \frac{|\delta \xi_{r0}|^2}{r_s^2(\gamma_L/\omega_0)^2}
\]

having made use of (\( \beta_{Er} \) is the \( \beta \) of the resonant energetic particles):

\[
\Im \delta W_{nk}^L \sim \frac{R_0}{r_s} \beta_{Er}
\]

Equating the two terms in order to have dominant kinetic nonlinearities (making use of \( |\Lambda| \approx |s| \), linear estimate) we get:

\[ \beta_{Er} \gg |s|^2 \frac{r_s}{R_0} \]
Linear dynamics

\[ f_{\text{distr}} \text{ superimposed on power } u(\theta=0,\mu): \]
\[ \text{caso}_34 f_{\text{distr}} (u,\mu), (u(\theta=0),\mu) \]
\[ \text{Caso-off-axis-24} f_{\text{distr}} (u,\mu) \]
Linear dynamics

- Typical simulation, linear phase
  \( n=1, m=1, 2, 3, 4, 5, 6 \)
Linear dynamics?

- Ho anche casi con $f_{\text{electron}}$ purely circulating ($u>0$, $u<0$), con i due profili di densità’ (on-axis, off-axis)
- per modi crescenti, il segno della frequenza di rotazione dipende solo dal segno di $\cos \alpha_0$, e non dal rapporto $\omega_*/\omega$: conta termine \propto \frac{dF}{dE}$ in QF?
- Comunque, non mostrerei questi casi, ho dato gia’ abbastanza spazio alla parte lineare (in particolare, con le slides che seguono, in cui mostro l’effetto del ritenere cineticamente:

1. solo elettroni energetici,
2. elettroni energetici + ioni termici (Landau damping e enhanced inertia)
3. elettroni energetici + ioni termici + elettroni termici

UNA NOTA SUL SEGNO DELLA FREQUENZA REALE: convenzione FFT in HMGC opposta a quella utilizzata in teoria analitica???
- teoria MHD di solito assume $\xi(r,t) = \xi(r) \exp(-i\omega t)$ (con frequenza complessa $\omega = \omega_R + i\gamma$) così da avere, per modi con frequenza reale nulla ($\omega_R=0$), con $\gamma$ positivo modi puramente crescenti.
- La frequenza ottenuta in HMGC, da FFT di $\phi(r,t)$, mi pare che usi una convenzione opposta:
  - nel caso di una simulazione in cui sia presente un singolo modo con frequenza $\omega_{\text{HMGC}}$, $\phi(r,t)=\phi_{\omega_{\text{HMGC}}}(r) \exp(+i\omega_{\text{HMGC}} t)$
  - Nelle simulazioni di HMGC, infatti, $\omega_{*he} >0$, serve $\omega >0$ per avere instabilita’, mentre la frequenza trovata con la FFT in HMGC e’ negativa…