Single-$n$ versus multiple-$n$ simulations of Alfvénic global modes

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Abstract: This work presents the results of a set of simulations of global Alfvén modes driven by an energetic particle (EP) population, with the specific aim of comparing single-$n$ and multiple-$n$ simulations ($n$ being the toroidal mode number). The hybrid reduced $O(\varepsilon_0^3)$ MHD gyrokinetic code HMGC is used, retaining both fluid (wave-wave) and energetic particles non-linearities ($\varepsilon_0$ being the inverse aspect ratio of the torus). Note also that HMGC retains self-consistently, in the time evolution, the wave structures as modified by the EP term. Simulations with the toroidal mode numbers $1 \leq n \leq 10$ have been considered. A circular, shifted magnetic-surface equilibrium has been considered, characterized by a large aspect ratio ($\varepsilon_0=0.1$) and a parabolic safety factor profile $q(r)=q_0+(q_a-q_0)(r/a)^2$ with $q_0=1.1$ and $q_a=1.9$.

A bulk ion density profile $n_i \propto 1/q^2$, in order to have the toroidal gap radially aligned has also been assumed. The equilibrium (initial) EP distribution function has been considered to be an isotropic Maxwellian, with a radial density profile $n_H=n_{H0}\exp(-19.53(1-\psi/\psi_0)^2)$, $T_H/T_{H0}=1$, $\rho_{H0}/a=0.01$, $v_{H0}/v_{A0}=1$, $m_H/m_i=2$ ($T_{H0}$, $\rho_{H0}$ and $v_{A0}$ are the on-axis EP temperature, Larmor radius and thermal velocity, respectively, and $\psi$ the poloidal flux function, with $\psi_0$ its on-axis value).

For the specific energetic particle drive considered ($n_{H0}/n_{i0}=1.75 \times 10^{-3}$), single-$n$ simulations are either stable ($n=1$), weakly unstable ($n=2,3$) or unstable ($n\geq4$), with $n=4, 5, 10$ exhibiting the larger growth-rates, while $4 \leq n \leq 7$ the largest saturated amplitudes. A variety of modes are observed (TAEs, upper and lower KTAEs, EPMs). Nevertheless, no appreciable global modification of the EP density profile is observed after saturation.

On the contrary, the multi-$n$ nonlinear simulation exhibits larger growth-rates and higher saturation amplitudes on all the toroidal spectral components considered, and, as a consequence, it results in a conspicuous broadening of the EP radial density profile at saturation, thus showing an enhanced radial transport w.r.t. the single-$n$ simulations. Non-linear coupling between different toroidal Fourier modes results both from the MHD terms and from the Energetic Particles.
• Thermal (core) plasma:
  – described by reduced $O(\varepsilon_0^3)$ visco-resistive MHD equations in the limit of $\beta=0$ ($\varepsilon_0 \equiv a/R_0$) ⇒ equilibria with shifted circular magnetic surfaces only can be investigated
  – MHD fields: $\psi, \phi$ (poloidal magnetic flux function and electrostatic potential)
• Energetic-ion population:
  – described by the non-linear gyrokinetic Vlasov equation, expanded up to order $O(\varepsilon)$ and $O(\varepsilon_B)$, with $\varepsilon \sim \rho_H/L_n$
    the gyrokinetic ordering parameter and $\varepsilon_B \sim \rho_H/L_B < \varepsilon$, in the $k_\perp \rho_H << 1$ limit (guiding-center approximation)
    energetic particle pressure: $\Pi_\perp, \Pi_\parallel$
  – fully retaining magnetic drift orbit widths
  – solved by particle-in-cell (PIC) techniques.
    $k_\perp$: perpendicular component of the wave vector
    $\rho_H$: energetic ion Larmor radius
    $L_n, L_B$: the equilibrium density and magnetic field scale lengths
• Toroidal coordinates system $(r, \theta, \varphi)$
Equilibrium:

- \( \varepsilon_0 \equiv a / R_0 = 0.1; \ T_H / T_{H_0} = 1, \ \rho_{H_0} / a = 0.01, \ \nu_{H_0} / \nu_A = 1, \ m_H / m_i = 2; \ n_{H_0} / n_{i0} = 1.75 \times 10^{-3} \)
- \( q(r) = q_0 + (q_a - q_0)(r/a)^2 \) with \( q_0 = 1.1 \) and \( q_a = 1.9 \)
- \( n_i \propto 1/q^2 \)
- \( n_H = n_{H_0} \exp(-19.53 (1-\psi/\psi_0)^2)) \); EP equilibrium distribution function \( F_{H,\text{eq}} \) is isotropic Maxwellian

\[ s = \sqrt{\left| \psi_{\text{eq}} - \psi_0 \right| / \left| \psi_{\text{edge}} - \psi_0 \right|} \]

\( \psi_{\text{eq}} \) the equilibrium magnetic poloidal flux function, and \( \psi_0 \) and \( \psi_{\text{edge}} \) its values, at the magnetic axis and at the edge, respectively.

**Fourier space** for perturbed quantities: \((m,n)\) and \((-m,-n)\) modes included in the simulations.
**Single-\( n \) simulations.** Toroidal mode numbers \( 1 \leq n \leq 10; \ n=0 \) not evolved; total (kinetic +magnetic) energy for each toroidal mode number \( n \); growth rate

\[
W_{\text{tot}} = W_{\text{kin}} + W_{\text{mag}}
\]

- \( n=1 \) is stable; \( n=4, 5, 10 \) have higher linear growth rate
- very weak broadening of the global (overall velocity space) EP radial density profile at saturation is observed in single-\( n \) simulation for \( n=4 \); broadening for \( 5 \leq n \leq 7 \) is barely appreciable.
Single-$n$ simulations. Toroidal mode numbers $1 \leq n \leq 10$, frequency spectra of the e.s. potential $\varphi(r, \omega)$, $t/\tau_{A0}=120$
Single-\(n\) simulations. Toroidal mode numbers \(1 \leq n \leq 10\), frequency spectra of the e.s. potential \(\varphi(r, \omega)\), \(t/\tau_{A0} = 240\)
**Single-\(n\) simulations.** Toroidal mode numbers \(1 \leq n \leq 10\), frequency spectra of the e.s. potential \(\varphi(r, \omega)\), \(t/\tau_A = 360\)
Single-n simulations. Toroidal mode numbers $1 \leq n \leq 10$, spectrograms of the e.s. potential $\phi(r,\omega)$
Single-$n$ simulations. Toroidal mode numbers $1 \leq n \leq 10$, linear eigenfunctions of the e.s. potential $\varphi_{m,n}(r)$, and wave-particle power exchange in $(u, \mu)$ plane ($u$: EP parallel velocity; $\mu$: EP magnetic moment)
Multiple-$n$ simulation

Toroidal mode numbers $1 \leq n \leq 10$

- **Standard** picture:
  1. strongest modes saturate first, because of non-linear (NL) energetic particle (EP) terms (e.g., flattening of EP radial density profile, at least for the resonant EP fraction);
  2. sub-dominant modes can, on turn, be driven unstable (or more unstable) because of the modifications to the EP distribution induced by the saturation of the dominant modes

- **Novel** observations from these set of simulations:
  1. NL mode-mode coupling from MHD terms, or mediated by EP term, strongly drives sub-dominant modes already during the linear growth phase of the dominant modes;
  2. sub-dominant modes driven non-linearly have field $(\psi, \phi)$ radial profiles and real frequencies substantially different from linearly unstable, single-$n$ modes;
  3. all the toroidal modes saturate almost simultaneously, inducing an enhanced EP transport (enhanced w.r.t. the single-$n$ simulations);
  4. On a longer time scale, after saturation of the faster modes, other subdominant modes can, in turn, be driven unstable (or more unstable) because of the modifications to the EP distribution (as #2 above, not investigated here…)
Multiple-\( n \) simulation. Toroidal mode numbers \( 1 \leq n \leq 10; \ n=0 \) not evolved

Stable or weakly instable low-\( n \) modes (\( n=1,2,3 \)) are non-linearly driven by dominant modes, during the linear growth phase of their time evolution
Multiple-\( n \) simulation. Frequency spectra of subdominant \( n \) are strongly modified w.r.t. the single-\( n \) simulations, \( \tau A_0 = 150 \).
**Multiple-$n$ simulation.** Eigenfunctions of subdominant $n$ are strongly modified w.r.t. the single-$n$ simulations, $t/\tau_{A0}=150$. 

10^{-15} \quad 10^{-13} \quad 10^{-11} \quad 10^{-9} \quad 10^{-7} \quad 10^{-5} \quad 0 \quad 50 \quad 100 \quad 150 \quad 200 \quad 250 \quad 300 \quad 350

- $n=1$
- $n=2$
- $n=3$
- $n=4$
- $n=5$
- $n=6$
- $n=7$
- $n=8$
- $n=9$
- $n=10$

- sum\_n1\_multiple\_n\_tot\_caso13tris
- sum\_n2\_multiple\_n\_tot\_caso13tris
- sum\_n3\_multiple\_n\_tot\_caso13tris
- sum\_n4\_multiple\_n\_tot\_caso13tris
- sum\_n5\_multiple\_n\_tot\_caso13tris
- sum\_n6\_multiple\_n\_tot\_caso13tris
- sum\_n7\_multiple\_n\_tot\_caso13tris
- sum\_n8\_multiple\_n\_tot\_caso13tris
- sum\_n9\_multiple\_n\_tot\_caso13tris
- sum\_n10\_multiple\_n\_tot\_caso13tris
Multiple-\( n \) simulation. Both MHD non-linearities and mode coupling through EP non-linearities are important.

- In the multiple-\( n \) simulation, EP drive only (blue curves) already gives NL coupling (see, e.g., the \( n=1 \) case, which, for the single-\( n \) simulation, i.e. with only EP drive as obtained by fluctuating fields with only \( n=1 \) components, is stable);
- in the multiple-\( n \) simulation with EP drive plus fluid non-linearities (red curves), fluid non-linearities anticipate a bit in time, without changing appreciably the growth-rate, the growing for the sub-dominant modes (not for the dominant one, \( n=4 \); the other stronger one, \( n=10 \), is almost unchanged during its linear phase, but it is non-linearly driven at higher overshooting after the first roll over), thus typically making the individual \( n \) components to overshoot more compared with the multiple-\( n \) simulation with only EP non-linearities.
**Multiple-\(n\) simulation.** EP radial transport enhanced w.r.t. single-\(n\) simulations when fluid+EP non-linearities are considered.

Multiple-\(n\) simulation including both MHD and EP NLs results in enhanced EP radial transport (w.r.t. single-\(n\) simulations).
Mode coupling through the EP term (1).

Hybrid reduced $O(\varepsilon_0^3)$ MHD equations (HMGC) (Briguglio et al., Phys. Plasmas 2, 3711 (1995); Wang et al., Phys. Plasmas 18, 052504 (2011)).

\[
\frac{\delta \psi}{\delta t} = \frac{R^2}{R_0} \nabla \psi \times \nabla \varphi \cdot \nabla U + \frac{B_0}{R_0} \frac{\partial U}{\partial \varphi} + \frac{c^2}{4\pi} \Delta^* \psi + O(\varepsilon^4 v_A B_0),
\]

\[
\dot{\rho} \left( \frac{D}{Dt} + \frac{2}{R_0} \frac{\partial U}{\partial Z} \right) \nabla^2 U + \nabla \left( \frac{D}{Dt} + \frac{1}{R_0} \frac{\partial U}{\partial Z} \right) \nabla U = \frac{1}{4\pi} \mathbf{B} \cdot \nabla \Delta^* \psi + \frac{1}{R_0} \nabla \left[ R^2 (\nabla P + \nabla \cdot \mathbf{\Pi}_H) \times \nabla \varphi \right]
\]

\[
+ O(\varepsilon^4 \rho \frac{v_A^2}{a^2}),
\]

\[
\Pi_s(t, x) = \frac{1}{m_s^2} \int d\bar{Z} D_{Z_s} \bar{Z} \bar{F}_s(t, \bar{R}, \bar{M}, \bar{V}) \times \left[ \frac{\Omega_s \bar{M}}{m_s} \mathbf{I} + \mathbf{b} \mathbf{b} \left( \bar{V}^2 - \frac{\Omega_s \bar{M}}{m_s} \right) \right] \delta (x - \bar{R})
\]

\[a_s = (e_s/c)(R_0/R)\psi; \quad U = -c\phi/B_0;\]

\[\psi\] is the magnetic stream function; \[\phi\] is the e.s. potential; “s” stay for EP species, thermal ions, ...

\[Z = (\mathbf{R}, M, V)\] are the gyrocenter coordinates, \[dZ/dt\] the phase-space velocities, \[(dZ/dt)_{\text{pert}}\] the perturbed ones; \[F_s;_{\text{eq}}\] the equilibrium distribution function of the “s” EP species.
Mode coupling through the EP term (2).

Mode coupling through the EP term \( \nabla \cdot \Pi_H \) means that a toroidal mode number “\( n \)” gets a contribution from quantities related to the EPs characterized by modes “\( n_1 \)” and “\( n_2 \)” such that:

\[
n = n_1 + n_2 \quad \text{(three waves scheme)}
\]

These kind of terms are indeed present, as can be recognized schematically by the following:

\[
\Pi_H \propto \delta F_H:
\]

\[
\left( \frac{\partial}{\partial t} + \frac{dZ^i}{dt} \frac{\partial}{\partial Z^i} \right) \delta \tilde{F}_H = - \left( \frac{dZ^i}{dt} \right)_{\text{pert}} \frac{\partial}{\partial Z^i} \tilde{F}_{H;\text{eq}}
\]

After formally splitting the generalized velocities in the l.h.s. in unperturbed “unpert” and perturbed “pert” ones:

\[
\left[ \frac{\partial}{\partial t} + \left( \frac{dZ^i}{dt} \right)_{\text{unpert}} \frac{\partial}{\partial Z^i} \right] \delta \tilde{F}_H = - \left( \frac{dZ^i}{dt} \right)_{\text{pert}} \frac{\partial}{\partial Z^i} \tilde{F}_{H;\text{eq}} - \left( \frac{dZ^i}{dt} \right)_{\text{pert}} \frac{\partial}{\partial Z^i} \delta \tilde{F}_H \quad \text{F.T.} \quad \sum_{n_1}(dZ^i/dt)_{n_1}
\]

And passing to toroidal Fourier components (equilibrium: “\( n=0 \)”; perturbed: “\( n \)”):

\[
\left[ \frac{\partial}{\partial t} + \left( \frac{dZ^i}{dt} \right)_0 \frac{\partial}{\partial Z^i} \right] \delta \tilde{F}_{H;n} = - \left( \frac{dZ^i}{dt} \right)_n \frac{\partial}{\partial Z^i} \tilde{F}_{H0} - \Sigma_{\tilde{n}} \left( \frac{dZ^i}{dt} \right)_{n-\tilde{n}} \frac{\partial}{\partial Z^i} \delta \tilde{F}_{H;\tilde{n}}
\]

From the last, convolution term, it can arise NL coupling between different \( n \)’s through the EP term.
Conclusions.

- Comparison between single-\(n\) and multiple-\(n\) simulations of Alfvénic modes has been performed, using the HMGC code; multiple-\(n\) simulations with the toroidal mode numbers \(1 \leq n \leq 10\) have been considered.
- In single-\(n\) simulations, the equilibrium considered (circular cross section, low inverse aspect ratio, \(\varepsilon_0 = 0.1\)), in presence of a Maxwellian EP population, result as either stable \((n=1)\), weakly unstable \((n=2, 3)\) or unstable \((n \geq 4)\), with \(n=4, 5, 10\) exhibiting the larger growth-rates; a variety of modes are observed (TAEs, upper and lower KTAEs, EPMs). Weak or negligible EPs radial transport is observed at saturation, for all the toroidal mode numbers considered.
- In multiple-\(n\) simulation, NL mode-mode coupling from MHD terms and mediated by EP term (three wave coupling), strongly drives sub-dominant modes already during the linear growth phase of the dominant modes; radial profiles of e.m. fields \((\psi, \phi)\) and real frequencies are substantially different from linearly unstable, single-\(n\) modes; all the toroidal modes saturate almost simultaneously, inducing enhanced EP transport (w.r.t. the single-\(n\) simulations).

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