NUMERICAL SIMULATIONS OF SAWTEETH IN TOKAMAKS

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ABSTRACT. Numerical simulations of sawteeth in tokamaks have been carried out using reduced magneto-hydrodynamics and a simple transport model. The electron temperature is evolved self-consistently, including Ohmic heating and a highly anisotropic thermal diffusivity. The sawtooth period and collapse time found in the simulations for Lundquist numbers $S$ below $10^7$ compare favourably with experimental results from small and medium size ohmically heated tokamaks. The sawtoothing is found to be sensitive to the values of the transport coefficients. In particular, the perpendicular viscosity must be comparable to, or larger than, the perpendicular thermal diffusivity for distinct relaxation oscillations to occur. To study the scaling with $S$, the viscosity and perpendicular thermal conductivity have been scaled as $1/S$, and $S$ has been varied. Modifications of the equilibrium, caused by the sawteeth, play an important role in the scaling of the collapse time and period with the Lundquist number. The self-consistently computed $q$-profiles are very flat in the central region where $q \approx 1$. Outside the low-shear region, the shear rises sharply. This $q$-profile allows a resistive mode to be turned on quickly with a high growth rate. The deviation of the central $q$ from unity over the sawtooth cycle decreases with increasing $S$, and the collapse time shows a weaker dependence than the $S^{1/2}$ scaling of Sweet and Parker.

1. INTRODUCTION

Sawtooth oscillations, in which the central temperature periodically shows a sudden drop followed by a slow increase until the next drop [1], occur in practically all tokamaks under a variety of experimental conditions. In 1975, Kadomtsev proposed a theoretical model for sawteeth [2]. In this model, the drop is triggered by the $m=1/n=1$ resistive kink mode, which becomes unstable when the safety factor in the centre falls below unity. Via resistive reconnection of the helical flux inside the original $q=1$ surface, the deformation relaxes to a symmetrical state with $q$ above unity everywhere and a flattened temperature profile in the central region of the plasma. After such an internal disruption, the temperature and current profiles peak again under the influence of Ohmic heating and resistive diffusion, making the central $q$ fall below unity, and the cycle is repeated. Recent experimental data indicate that the Kadomtsev model is not always applicable and that the central $q$ may be significantly less than unity while the discharge is still sawtoothing [3]. Furthermore, measurements on JET indicate that the growth time of the instability leading to the drop in central temperature is too short to be connected with a resistive mode [4, 5].

The purpose of the present study has been to explore, by means of self-consistent numerical simulation and a systematic parameter study, the predictions of the reduced magnetohydrodynamic (MHD) model for the sawteeth and to compare the simulation results with experimental observations. Waddell et al. [6] first simulated a single sawtooth crash, using reduced MHD. Sykes and Wesson [7] studied repeated oscillations, assuming Spitzer resistivity, $\eta \propto T^{-3/2}$, and using an equation for the temperature evolution that included Ohmic heating and perpendicular thermal diffusion. In this model, the oscillations were found to decay in time. Denton et al. [8, 9] and Bondeson [10] reproduced periodic oscillations by introducing a large thermal conductivity along the field lines.

All computations carried out so far have been made with plasma parameters far from those characteristic of current experiments. In the present paper, we report results of self-consistent simulations performed with parameters representative of small and medium size
tokamaks and compare the results with experimental data. Our principal finding is that the reduced MHD simulations reproduce surprisingly well the behaviour of Ohmic discharges with Lundquist numbers $S$ (ratio of resistive diffusion time to Alfvén time) up to about $10^7$, for example with respect to the sawtooth period and collapse time. In particular, the collapse time shows a weaker dependence on resistivity than the $S^{1/2}$ dependence expected from the Sweet–Parker scaling [11, 12], mainly because the change in central $q$ over the sawtooth cycle decreases with increasing $S$. Consequently, the amount of helical flux to be reconnected decreases with increasing conductivity and this partly compensates for the decrease in reconnection rate. In our self-consistent simulations, the maximum growth rate of the resistive kink mode scales weakly with $S$, roughly as $S^{-1/3}$. The weak dependence of the growth rate on $S$ and the rapid turn-on of the precursor oscillation result from the modification of the equilibrium profiles by the sawteeth themselves. In the self-consistently computed equilibria, the shear changes from nearly zero inside a central region to order unity over a distance comparable to a resistive layer width. The sawtooth period shows a weaker than linear dependence on $S$, in agreement with experimental results.

In carrying out these simulations, we have found that the sawtoothing is sensitive to the values of the transport coefficients, in particular the ratio of perpendicular viscosity $\nu$ to perpendicular heat diffusivity $\chi_\perp$. If $\nu/\chi_\perp$ is too small, the characteristic relaxation oscillations of the sawteeth are replaced by more or less continuous mode activity and the equilibrium never departs significantly from marginal stability. However, if $\nu > \chi_\perp$, distinct relaxation oscillations occur. In this respect, the primary effect of viscosity is to damp the successor oscillations. If the damping is too weak, the successor oscillations do not decay sufficiently before the next crash is triggered, and continuous mode activity results. Recent experimental results concerning momentum confinement in Doublet III [13], TFTR [14] and ASDEX [15] all indicate that the diffusivity of momentum is of the same order as that of thermal energy, although both are anomalous.

### 2. THE MODEL

Our simulations are based on the standard, low beta, resistive, reduced MHD equations, derived in the large aspect ratio, cylindrical limit with $\beta = O(\epsilon^2)$ [16]. The code [10] evolves the electron temperature self-consistently with a highly anisotropic thermal diffusivity and Ohmic heating. In normalized units, the model equations are:

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v}\right) \omega = \mathbf{B} \cdot \nabla j + \nu \nabla^2 \omega$$

$$\frac{\partial \psi}{\partial t} = \mathbf{B} \cdot \nabla \phi - \eta (j - j_0) + E_0(t)$$

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v}\right) T = \frac{2\eta}{3} (j - j_0) + \mathbf{B} \cdot \nabla \chi_\perp (\mathbf{B} \cdot \nabla) T$$

Here, $\mathbf{B} = \nabla \psi \times \mathbf{z} + B_r \mathbf{z}$, $\psi$ is the magnetic flux function, $\nabla = \nabla \phi \times \mathbf{z}$, and $\phi$ is the stream function, $\omega = -\nabla^2 \phi$ is the vorticity, $j = -\nabla^2 \psi$ is the plasma current, $T$ is the electron temperature, $\eta$ is the Spitzer resistivity and $\chi_\perp = \nabla \cdot \mathbf{z} \partial \phi / \partial x$. The unit length is the minor radius $a$, the unit time is the Alfvén transit time, $T_A = R/v_A$, where $R$ is the major radius and $v_A$ is the Alfvén speed in the toroidal magnetic field $B_t$. $\nu$ is the perpendicular viscosity and $\chi_\perp$ is the perpendicular thermal diffusivity, both multiplied by $R/a^2$, and $\chi_\parallel$ is the parallel thermal diffusivity multiplied by $a^2/v_A^2$. In Eqs (1), the mass density, viscosity and parallel thermal conductivity are taken to be constant in space and time.

The normalized temperature $T$ corresponds to the poloidal beta of the electrons divided by $q_i^2$:

$$T = (R/a)^2 \beta / 2.$$  

In the following, we refer to the central value of $T$ as $\beta_{pol}$. It is proportional to the ratio of energy confinement time, $\tau_E = 3n\tau T/2\eta^2$, to resistive diffusion time, $\tau_R = a^2/\nu \eta$. To specify the resistivity of a given simulation, we quote the Lundquist number, $S = \tau_R/\tau_A$, where $\tau_R$ is evaluated by using the central resistivity.

The code uses finite differences in the radial direction and Fourier expansion in the azimuthal and toroidal directions. In the present study, we have used 200 non-equidistant radial points and, in most cases, considered single-helicity perturbations with $m/n = 1$. Mode numbers up to $m = 4$ (in some cases $m = 8$) have been retained. No assumption is made regarding the symmetry of the perturbations, i.e. $\psi$, $\phi$ and $T$ are expanded with both cosine and sine components. The inductive toroidal electric field $E_0(t)$ is adjusted in time to keep the $q$-value at the edge fixed.

We have chosen as a reference case a typical Ohmic discharge in the Frascati Tokamak (FT), with
BT = 6 T, a = 0.20 m, R = 0.83 m, qa = 2.6, central density n(0) = 1.8 × 10^{20} m^{-3} and central temperature between 500 and 1000 eV. With these parameters, the Alfvén time is \( \tau_A = 0.085 \mu s \), the Lundquist number is \( S = 10^7 \), the normalized parallel thermal conductivity is \( \chi_\parallel = 20 \) and the normalized temperature is \( T = 0.02-0.04 \). We assume a radial dependence for the perpendicular thermal diffusion coefficient, \( \chi_\perp(r) = \chi_\perp(0)/[1 - (r/r_0)^2]^2 \), with \( r_0/a = 1.1 \), to represent results from power balance studies on FT.

Although Eqs (1) neglect toroidal and finite pressure effects for the MHD perturbations, we have, in most cases, kept two neoclassical effects: the bootstrap current \( j_{bs} \) and trapped particle corrections to the resistivity \( [17] \). The neoclassical corrections to the resistivity have significant effects on the current profile, while the bootstrap current is small for the beta values achieved in Ohmic tokamaks. The neoclassical resistivity at the \( q = 1 \) radius (\( \approx 0.45a \)) is typically about twice the central value in our simulations.

In the present discussion of the sawteeth, where toroidal and finite pressure effects are looked for to explain experimental results, as in Refs [3, 4], the use of a cylindrical zero-beta approximation such as Eq. (1) may be considered questionable. We share this point of view. On the other hand, there are nonetheless good reasons for carrying out reduced MHD simulations. First of all, self-consistent simulations of sawtoothing have previously not been performed with parameters close to experimental values and, consequently, the predictions of the cylindrical approximation for actual tokamak experiments are not really known. Since, in addition, measurements of fundamental quantities, such as central \( q \), have given significantly different results in different experiments [3, 18, 19], it is, ironically, not well understood at present to what extent the cylindrical approximation agrees or disagrees with experimental results. The aim of our investigation has been to establish more firmly the degree to which the cylindrical model reproduces experimental results concerning the sawteeth, using as realistic a model as possible for the transport coefficients. Furthermore, by running a numerically efficient reduced MHD code [10], we have been able to undertake an extensive parameter study. This study reveals several unexpected and non-trivial dependences of the sawteeth on the transport coefficients. It is, however, evident that the results of a reduced MHD simulation must be interpreted with due caution.

### 3. DEPENDENCE ON THERMAL DIFFUSIVITY AND VISCOSITY

In abstract terms, sawtoothing can be thought of as a driven, non-linear, dissipative system in which relaxation oscillations occur. The behaviour of such non-linear dynamic systems is often sensitive to the parameter values, in particular to the dissipation coefficients. We have observed that this is indeed the case for sawteeth: exploratory simulations, performed with somewhat randomly chosen parameters, \( \eta, \nu, \chi_\perp \) and \( \chi_\parallel \), would produce very different results, ranging from distinct sawteeth to weak, continuous mode activity.
To obtain some understanding of the influence of the parameters on the character of sawteeth, we first studied the dependence on \( \nu \) and \( \lambda \), at moderate \( S \). We emphasize that the scaling studies presented here are self-consistent, taking full account of modifications of the equilibrium profiles resulting from the action of the sawteeth themselves. Such modifications lead to important and unexpected results for the scaling with the Lundquist number.

Figure 1 shows the time evolution of the central temperature \( T_0 \) and the safety factor \( q_0 \), together with energies in the Fourier components 1/1 and 4/4 (integrated over the cross-section) for three simulations at different values of \( \beta_{\text{pol}} \): 0.022, 0.15 and 1.35. For all three cases, \( S = 10^4 \) and \( \nu = 10^{-4} \). (For the two cases at high beta in Fig. 1, \( \beta_{\text{pol}} = 0.15 \) and \( \beta_{\text{pol}} = 1.35 \), we have discarded the neoclassical corrections to the resistivity and bootstrap current so that a direct comparison with simulations published in the literature [8] is possible.) Figure 1 shows clearly that as \( \beta_{\text{pol}} \) is reduced by increasing \( \lambda \) (i.e. by decreasing the energy confinement time), the period of the sawteeth decreases and the characteristic relaxation oscillations are lost. Pronounced relaxation oscillations occur in the high beta case, as can be seen from the 1/1 mode energy in Fig. 1(c). For lower beta, the sawtooth period is shorter and the successor oscillations do not have sufficient time to decay before the next crash occurs. In the three cases of Fig. 1, \( \chi_L(0) \) takes the values \( 1 \times 10^{-3}, 7 \times 10^{-5} \) and \( 7 \times 10^{-6} \), which may be compared with the viscosity, \( \nu = 10^{-4} \). The simulation with \( \beta_{\text{pol}} = 1.35 \) corresponds closely to that of Denton et al. [8]. In this case, the variation in \( q_0 \) over the sawtooth cycle is large, \( \Delta q_0 = 0.25 \).

If we try to represent the sawtooth period of the three cases as a power law, \( \tau_{\text{saw}} \propto \beta_{\text{pol}}^n \), the depen-
dence is rather strong, \( \alpha_\beta \approx 0.8 \). Thus the energy confinement time plays an important role in determining the period of sawteeth.

In FT, \( \beta_{pol} \approx 0.02 \), and the corresponding case (a) in Fig. 1 does not show regular sawteeth. However, the behaviour of the sawteeth is also affected by viscosity. Figure 2 shows the time evolution of \( T_0 \) and \( q_0 \), and the mode energies for four different values of viscosity: \( 10^{-3}, 10^{-4}, 10^{-5} \) and \( 10^{-6} \). The resistivity and the perpendicular heat conductivity are a factor of 20 larger than the reference case: \( S = 5 \times 10^5, \chi_\perp(0) = 2 \times 10^{-5} \). Distinct relaxation oscillations occur for \( \nu = 10^{-3} \) and \( 10^{-4} \). However, for \( \nu \leq 10^{-5} \), the activity becomes irregular, which is best seen from the plot of \( q_0(t) \). The period of the sawteeth increases with viscosity, as shown in Fig. 3, from \( \tau_{saw} = 1 \times 10^3 \tau_A \) at \( \nu = 10^{-6} \) to \( \tau_{saw} = 3.3 \times 10^3 \tau_A \) at \( \nu = 10^{-3} \). For \( \nu \geq 10^{-4} \), we find \( \tau_{saw} \propto \nu^{\alpha_\nu} \) with \( \alpha_\nu \approx 0.3 \).

The reason for the irregular behaviour when \( \nu \leq 10^{-3} \) (which is similar to the low beta case in Fig. 1) can be understood from the time evolution of the mode energies. The variation in the 1/1 mode energy is close to four orders of magnitude when \( \nu = 10^{-3} \) and less than one order of magnitude when \( \nu = 10^{-4} \). Apparently, an important role of the viscosity is to control the decay of the successor oscillations by changing the damping rate. The damping rate of the energy in the 1/1 mode is about \( 10^3 \) for \( \nu = 10^{-3} \) and \( 5 \times 10^3 \) for \( \nu = 10^{-6} \). Thus, the damping rate depends weakly on \( \nu \), but this has a strong (exponential) influence on the sawtooothing, in particular as the sawtooth period also increases with viscosity.

When the viscosity is sufficiently large, it reduces the growth rate of the resistive kink mode; asymptotically for \( \nu \gg \eta, \gamma = \gamma_{\nu=0} = 1.53 \left( \eta/\nu \right)^{1/3} \). This reduction of the growth rate also tends to make the period increase with viscosity.

We conclude that the three transport coefficients \( \chi_\perp, \nu \) and \( \eta \) affect the sawteeth in very different ways. Viscosity increases the damping of the successor oscillations and slows down the growth of the precursor oscillation. As a consequence, increasing \( \nu \) gives a longer sawtooth period and more pronounced relaxation oscillations. Increasing \( \beta_{pol} \) at fixed \( \eta \) gives a longer sawtooth period as the Ohmic heating time increases. Since \( \chi_\perp \) does not significantly influence the damping of the successor oscillations, a larger \( \beta_{pol} \) (smaller \( \chi_\perp \)) also gives more pronounced relaxation oscillations. We note that the experimental relation \( \gamma = \chi_\perp \) lies within, but not far from, the boundary of the region that produces sawtooth-like oscillations in Figs 1 and 2.

The influence of resistivity is more complex. At fixed \( \beta_{pol} \) and \( \nu \), the sawtooth period increases with \( S \), but this is accompanied by a decrease in the damping and growth rates. The increase in sawtooth period is stronger and the net result is the formation of clearer relaxation oscillations at large \( S \). This can be seen by comparing Figs 1(a) and 2(b), where \( S \) changes from \( 1 \times 10^4 \) to \( 5 \times 10^5 \), while \( \beta_{pol} \) and \( \nu \) are fixed. On the other hand, if \( \nu \) is scaled in proportion to \( \chi_\perp/\beta_{pol} \), as suggested by experiments, the character of the sawteeth at fixed \( \beta_{pol} \propto \eta/\chi_\perp \) shows a very weak net dependence on \( S \), as discussed in Section 4.

In addition to the dependence on \( \beta_{pol} \) and \( \nu \) discussed here, the sawteeth are sensitive to the parallel thermal conductivity. Too small a value of \( \chi_\parallel \) will completely eliminate the sawteeth and give rise to a steady \( m=1/n=1 \) convection pattern \([9, 10]\), which may be thought of as a non-linear form of the rippling mode. From our parameter study at low \( S \), it appears that the threshold value of \( \chi_\parallel \) for regular sawteeth to occur is approximately that which leads to a decay of the temperature perturbation in the \( q \approx 1 \) region by one order of magnitude between two successive crashes. In simulations, at large \( S \) \(( \geq 10^5 \), the rippling mode slightly influences the sawteeth by speeding up the linear growth when the central \( q \) is very close to unity. For the range of parameters that we have explored, the rippling modes have an insignificant effect non-linearly and no major change in the behaviour of the sawteeth occurs when the \( (m, n) \neq (0,0) \) components of the resistivity are turned off. The normalized parallel conductivity for FT from

\[
\frac{\tau_{saw}}{\tau_O} = \frac{S}{5 \times 10^4} \text{ versus viscosity } \nu \\
\nu = 0.3, 0.5, 0.7, 0.9 \\
S = 5 \times 10^5
\]
4. DEPENDENCE ON LUNDQUIST NUMBER

As shown in Section 3, the sawteeth are sensitive to the transport coefficients $\chi$, $\nu$ and $\eta$. It would be desirable to make a complete parameter study (at least three-dimensional), but this would be very costly in computer time. Instead, we have taken the point of view that in comparing ohmically heated tokamaks of different size, the primary variations are those in $S$, while $\nu S$ and $\chi S$ remain relatively constant.

We therefore consider the reference parameters for FT quoted in Section 2 and scale these by a common enhancement factor $E$ for all small transport coefficients: $\chi$, $\nu$ and $\eta$. Our self-consistent simulations show that the growth rate of the resistive kink mode and the collapse time are strongly influenced by the particular form of equilibrium profiles that occur as a result of sawteeth and that the modifications of the equilibrium profiles depend on $S$.

4.1. Simulation results

Figure 4 shows the sawtooth period as a function of $S$ (with $\nu S$ and $\chi S$ held constant) for our reduced MHD simulations, together with experimental data points for Ohmic discharges [4, 21-26]. To account for the dependence on $\beta_{\text{pol}}$, we have applied the approximate scaling of Section 3, $\tau_{\text{saw}} \propto \beta_{\text{pol}}^{0.8} S^{0.3}$, with $\nu \propto \chi \propto \beta_{\text{pol}}^{0.3}$. This implies that, for fixed $S$, the net dependence on $\beta_{\text{pol}}$ is $\tau_{\text{saw}} \propto \beta_{\text{pol}}^{0.5}$. Therefore, the experimental points have been plotted as $\tau_{\text{saw}}/\tau_{A}/\beta_{\text{pol}}^{0.5}$. (In comparing data for Ohmic discharges, the exponent for $\beta_{\text{pol}}$ is not very sensitive, since $\beta_{\text{pol}}$ does not vary much between different machines.) The agreement in Fig. 4 is striking; the simulation results fall within the scatter of the experimental data for $S \leq 10^6$; at $S = 10^7$, the period is somewhat too short in comparison with FT, for example. We conclude from Fig. 4 that with $\tau_{\text{sp}}/\tau_{A}$ fixed, $\tau_{\text{saw}}/\tau_{A} \propto S^\alpha$, with $\alpha = 0.7$.

We now discuss in more detail the results for various enhancement factors, $E = 25$, 5 and 1. Figure 5 shows the time histories of the central temperature and safety factor together with the mode energies in the three cases. It is clear that the variations in $q_0$ and $T_0$ diminish as $S$ increases. For example, at $S = 4 \times 10^5$, we have $\Delta q_0 \approx 0.05$ and $\Delta T_0/T_0 = 40\%$, whereas, at $S = 10^7$, we find $\Delta q_0 = 0.009$ and $\Delta T_0/T_0 = 15\%$. This is consistent with the fact that $\tau_{\text{saw}}$ shows a weaker than linear dependence on $S$.

It has been noted [4] that the collapse time in JET is short (~200 $\mu$s), seemingly at variance with the resistive reconnection rates and also with the assumption that a resistive mode triggers the internal disruption [5]. Instead, ideal instability has been suggested as a likely candidate for the trigger [27, 28]. While this conclusion may be correct for a machine of the size of JET, where $S$ is between $10^8$ and $10^9$, our simulations show that sawteeth caused by the resistive kink mode are in excellent agreement with experimental data for $S$ up to about $10^7$.

Figure 6 shows the collapse time (i.e. the time-scale over which $T_0$ drops) and the growth time of the linear instability from the non-linear simulations, for the three values of $E$. Notably, both of these times scale weakly with $S$, and at $S = 10^7$ they are in good agreement with observations on FT. Figure 6 indicates that the maximum linear growth rate scales approximately as $S^{-1/3}$. This would be in line with the prediction of conventional linear theory, if the shear at the $q=1$ surface were independent of $S$. Given that $dq_0/dt$ scales as $1/S$ and that the resistive kink mode becomes unstable as soon as $q < 1$ anywhere, the $S^{-1/3}$ scaling indicates that the shear, $s = r q'/q$, is not a fixed (i.e. $S$-independent) function of $r-r_{\text{sp}}$. We find that this is indeed the case by comparing the $q$-profiles in the three cases of $E = 25$, 5 and 1. Detailed examination of the $q$-profiles also shows that the shear varies considerably within the resistive layer. At the time of
maximum growth rate for the case with $S = 2 \times 10^6$, $s$ is about 0.2 at $q = 1$, and goes from 0.1 one-layer width inside the resonant surface to 0.3 one-layer width outside. At $S = 10^7$, the profiles have steepened, and the variation of the shear across the layer is about the same, even though the width of the resistive layer shrinks with increasing $S$. Thus, the current gradients become increasingly steep at the edge of the low-shear region as $S$ increases so that the shear in the resistive layer is almost independent of the Lundquist number.

Figure 7 shows the $q$-profile together with the $m=n=0$ current density for $E = 5$ ($S = 2 \times 10^6$) at the time of maximum growth rate, for the $m = 1$ component (a) and at the end of the reconnection phase (c). The sawteeth maintain the steep current profile by generating a sharp current dip just outside the low-shear region during the non-linear reconnec-

**FIG. 5.** Time evolution of the central temperature $T_0$, the safety factor $q_0$, and the integrated energies $W$ in the 1/1 and 4/4 Fourier components, for three different values of the Lundquist number $S = 10^7/E$, with $E = 25$, $E = 5$, $E = 1$. The other parameters are the same as in Fig. 4.

**FIG. 6.** Growth time of the linear instability (open circles) and collapse time of the central temperature (crosses; for each value of $S$, the minimum and maximum values as observed during several sawteeth are shown). The other parameters are the same as in Fig. 5.
4.2. Linear stability — comparison with toroidal results

To study the linear evolution of the precursor oscillation, we have computed the linear $m = 1$ growth rate as a function of $q_{\text{min}}$ for the current density profile in Fig. 7(a) by simply rescaling the current. This is of interest when we wish to see how quickly the instability can be turned on and enables a comparison with toroidal studies, including the effects of finite pressure [28]. The result for $S = 2 \times 10^6$ is shown in Fig. 8. Note that a change of less than $2 \times 10^{-3}$ in $q_{\text{min}}$ produces a growth rate $\gamma$ of $10^{-3} \tau_A$, i.e. the turn-on of instability is very rapid. We have also computed the $m = 1$ linear growth rates for the sequence of equilibria obtained in the simulations, from the time of marginal stability to the time $1000 \tau_A$ later, when the growth rate is about $2 \times 10^{-3} \tau_A^{-1}$. The dependence of $\gamma$ on $q_{\text{min}}$ is practically identical with that shown in Fig. 8. However, at the marginal point for this sequence of equilibria, $q$ is flatter in the central region: $1.002 < q(r) < 1.003$ for $r < 0.42$.

One feature is immediately apparent in Fig. 8: the marginal $q_{\text{min}}$ is slightly above unity, by about $2 \times 10^{-3}$. The mode that is unstable when $q_{\text{min}} > 1$ is evidently destabilized by rippling, i.e. by perturbations in the resistivity. With the resistivity perturbations removed, the marginal $q_{\text{min}}$ is almost exactly 1, and the
growth rates are somewhat lower than those shown in Fig. 8.

The sensitivity of the MHD stability properties to the equilibrium profiles in the \( q = 1 \) region makes a precise comparison with the toroidal results given in Ref. [28] somewhat difficult. Among the \( q \)-profiles studied in Ref. [28], the one which comes closest to that shown in Fig. 7(b) is the 'ultra-flat' profile, \( q(r) = q_0 \left[ 1 + \left( r/r_0 \right)^2 \right]^{1/\lambda} \), with \( \lambda = 6 \). Figure 15 of Ref. [28] gives growth rates for the ultra-flat profile as functions of \( q_0 \) for different values of \( \beta \) and at tight aspect ratio, \( A = 2.5 \). We first compare the case of \( \beta = 0 \) in Ref. [28] with our results presented in Fig. 8. The marginal point is \( q_{\text{min}} \approx 1 \) in both cases, but the growth rate shown in Ref. [28] is lower than that in our Fig. 8. (In the numerical examples of Ref. [28], \( S \) is defined with respect to the minor radius \( r_0 \) and therefore the resistivity at the \( q = 1 \) surface is about the same in the two cases.) As already discussed, the resistive kink mode dispersion relation (which is somewhat oversimplifying for the types of profile under consideration) predicts a growth rate proportional to \( s_{\text{eq}}^2 \) and we see that this varies very quickly with \( q_{\text{min}} \) for the types of profile occurring in the self-consistent simulations. For example, for the profile in Fig. 7(b), \( s = 0.1 \) is reached already with \( 1 - q_{\text{min}} = 0.003 \). The ultra-flat profile, for which \( s_{\text{eq}} = 2 \left( 1 - q_0^2 \right) \), needs \( 1 - q_{\text{min}} = 0.0085 \) to reach the same shear. It is clear that the strong variation of the equilibrium over a region comparable to the resistive layer permits a rapid turn-on of the resistive mode at high \( S \). Thus, even local changes in the equilibrium profiles are important for the scaling of the linear growth rate with \( S \).

Figure 8 refers to the equilibrium computed at \( S = 2 \times 10^6 \). For larger \( S \), the equilibrium profile becomes even steeper near the minimum \( q \). In the non-linear simulations, where the equilibrium depends on \( S \) in a self-consistent manner, and with resistivity perturbations included, the net dependence of the maximum 1/1 growth rate over the sawtooth cycle in the non-linear simulations is rather weak, approximately \( \propto S^{-1/3} \), as shown in Fig. 6.

It should be stressed that although the linear growth is sped up by the resistivity perturbations, these have little effect on the non-linear evolution, for which we find that the resistive kink mode plays the dominant role. Almost no change occurs in the non-linear simulations at \( S = 2 \times 10^6 \) when all the resistivity components except \( m=n=0 \) are turned off; however, at \( S = 10^7 \), the sawtooth period is increased by about 20% (which actually brings the simulations closer to the experimental result, see Fig. 4).

In comparison with toroidal calculations, it is clear that a significant omission in our model is the neglect of the plasma pressure. For the profiles used in Figs 14 and 15 of Ref. [28], a beta of 1% gives rise to an ideal instability at \( q_0 \) slightly above 1, with growth rates somewhat larger than those of the resistive mode shown in Fig. 8. However, it should be noted that the increase in \( q_0 \) at marginal stability is only about 0.01 and that the volume average beta in most ohmically heated tokamaks is only a few tenths of a per cent. In addition, the pressure profiles are often less peaked than the one used in Ref. [28], which further reduces the destabilizing pressure gradient in the central region. Thus, Fig. 15 of Ref. [28] refers to a case of significantly higher beta than in Ohmic tokamaks. This indicates that the finite beta effects are small in ohmically heated tokamaks at \( S = 10^6 \), although, for \( q_{\text{min}} \) slightly above unity, triggering can occur when finite pressure is accounted for. It is noteworthy that in our simulations the resistivity perturbations have a similar effect.

The successful reproduction of experimental results for crash times and sawtooth periods by our reduced MHD simulations together with the comparison between the cylindrical and toroidal results for this type of \( q \)-profile gives strong evidence that the resistive kink mode plays a dominant role as a trigger of the sawteeth in Ohmic discharges with moderate \( S \) \((S < 10^7)\). For theoretical reasons, at higher \( S \), it must be expected that toroidal modes are involved. This is not in contradiction to the results of our simulation study, which fails to reproduce the experimental behaviour for \( S > 10^7 \), for example, by giving too
short a period. It therefore seems probable that there is a continuous transition between sawteeth triggered by an essentially cylindrical resistive kink mode at low S and sawteeth triggered by a toroidal (and possibly ideal) mode at larger S, with the transition occurring at Lundquist numbers of the order $10^7$. Needless to say, our cylindrical simulations cannot describe sawteeth in which the central q remains below unity over the entire cycle.

4.3. Non-linear evolution

Similar to the linear growth time, the collapse time also shows a weak dependence on S (see Fig. 6) — much weaker than the $S^{1/2}$ dependence expected from the Sweet-Parker scaling [11, 12]. This can be understood because the variation in $q_0$ decreases with increasing S and, therefore, also the helical flux to be reconnected decreases (see Fig. 5). Since the proposal of the Sweet-Parker scaling in 1957/1958, it has been realized that higher reconnection rates are possible, depending on global conditions [31]. The physical situation envisaged by Sweet and Parker was a symmetrical current sheet. This is quite different from the current sheet relevant to the present model of the sawtooth crash, where the conditions are entirely different on the two sides of the $q = 1$ surface; inside, the helical field to be reconnected is small (and decreases when S increases), but, on the outside, the shear takes off very sharply. The helical Alfvén frequency inside the low shear region, $\propto |q_0 - 1|$, need not be determining for the reconnection rate, since the rigid $m = 1$ displacement of the interior is not an Alfvén wave but rather a fast magnetosonic wave. (In the reduced MHD theory, the fast wave exists as a 'surface wave', with $\omega = -\nabla \phi = 0$, analogous to pressure perturbations in an incompressible fluid. For $m = 1$, the 'surface wave' is a rigid displacement of the core, due to 'forces' $B \cdot \nabla$ acting at its boundary.) The effective ideal time for non-linear reconnection appears to be determined by the shear just outside the low-shear region rather than the Alfvén time in the centre. This is in analogy with linear theory. The weak variation of the reconnection time with S seen in Fig. 6 is in sharp

![Graphs showing time evolution of central temperature and mode energies](image)

**FIG. 9.** Time evolution of the central temperature $T_0$ and the mode energies $W$ for the last crash of Fig. 5 ($S = 2 \times 10^6$).

(a) and (b) Simulation with Fourier components up to $m/n = 4/4$.

(c) and (d) Simulation with Fourier components up to $m/n = 8/8$. 

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disagreement with the Sweet-Parker scaling when the effective ideal time is taken as $\tau_A |q_0 - 1|^{-1}$. We plan to address this issue in more detail in the future.

Figure 5 shows that during the non-linear part of the internal disruption the $m=4/n=4$ mode is excited to a rather high amplitude, and the question arises whether our simulations with $m \leq 4$ have sufficient angular resolution. To check this, we have rerun the case with $S = 2 \times 10^6$, keeping Fourier components up to $m=8/n=8$. The result for $T_0(t)$ is shown in Fig. 9. When examining the gross features, such as $T_0(t)$, the four Fourier components give adequate resolution. However, when we consider details of the evolution, certain differences become apparent, notably the formation of small secondary islands along the $q = 1$ separatrix [32] which are seen in the simulation with eight modes but not in that with four modes. These details appear to have only very slight influence on the global evolution. Figure 10 presents contour plots of the helical flux function, $\psi = \psi - (1 - \tau^2)/2$, at various stages of the internal disruption for the eight-mode simulation at $S = 2 \times 10^6$. The formation of the small secondary island is followed by a coalescence with the primary island. It is easily seen that components with $m > 2$ are excited at rather high amplitude, as is evident also in the contour plots of the temperature (Fig. 11). We note from the temperature plots that the shape of the hot core changes during the internal disruption from a circle to a crescent shape, as was observed in JET [33] and TFTR [34].

Another issue of resolution concerns the influence of multiple helicity interactions. For example, the very steep current profile in Fig. 7(c) might make tearing modes with nearby rational surfaces unstable. This was predicted theoretically by Kadomtsev [2] and was observed in a multiple helicity simulation at moderate $S$ [10] where the $m/n = 4/3, 5/4, 6/5$, etc. modes were periodically destabilized by the sawteeth.
We have investigated this by including modes with \( m = n + 1 \) in addition to the \( m = n \) modes with \( 0 \leq n \leq 8 \). Although the \( m = n + 1 \) modes are periodically destabilized, they do not reach a sufficient amplitude, even at \( S = 10^7 \), to have a significant effect on sawtooothing. (This result may be quite different in toroidal geometry because of toroidal coupling.)

5. SCALING OF THE SAWTOOTH PERIOD

Our simulations predict that the sawtooth period scales as \( \tau_{saw} \propto \beta_{pol}^{0.5} S^{0.7} \), which, as shown in Fig. 4, gives a good fit for different machines. For a comparison with scalings observable in a single machine, it is useful to display the dependence on the plasma parameters: the density \( n \), the temperature \( T \), \( Z_{eff} \), the ratio between the ion mass and the proton mass \( A \), the toroidal field \( B_T \) and the linear dimensions. From \( \tau_{saw} \propto \beta_{pol}^{0.5} S^{0.7} \), we obtain (for fixed \( q_a \)):

\[
\tau_{saw} \propto (R/B_T)^{1.3} A^{0.15} n^{0.65} T^{1.35} \quad (2a)
\]

Here,

\[
f(Z_{eff}) = Z_{eff} [0.29 + 0.457/(1.077 + Z_{eff})] \quad (2b)
\]

is the \( Z_{eff} \) dependence of the resistivity. When the density is changed at fixed current, the temperature will change, and Eq. (2), when combined with the relations \( T = T(n) \) and \( Z_{eff} = Z_{eff}(n) \), gives a prediction for the sawtooth period as a function of the electron density. The scaling (2) is in reasonable agreement with results from FT [35], where \( \tau_{saw} \propto n^{0.5} \) is observed together with a weak inverse dependence of the temperature on the density. We note that FT has an unusually small fraction of impurities, and \( Z_{eff} \approx 1 \). In other machines, the sawtooth period has a stronger dependence on density; in TFR [36] and TCA [26], for example, \( \tau_{saw} \) increases approximately linearly with \( n \). It is probable that part of this dependence is due to
changes in $Z_{eff}$; an inverse dependence, $Z_{eff} \propto 1/n$, would be more than sufficient to bring the scaling (2) into agreement with the TFR and TCA results.

6. DISCUSSION

We have simulated the sawtooth activity of ohmically heated tokamaks using a reduced MHD code with transport. The sawteeth have been found to be sensitive to the transport coefficients. In particular, the perpendicular viscosity must be of the same order as, or larger than, the perpendicular heat diffusivity for regular sawtoothing to occur. In the reduced MHD model, the sawtooth collapse occurs via a resistive kink mode and complete resistive reconnection. Our main conclusion is that, despite its simplicity, the model is in excellent agreement with experimental results as regards the sawtooth period and crash times for $S$-values up to about $10^7$.

The self-consistent simulations produce q-profiles that are very flat in the central region where $q \approx 1$. At the edge of this region, the current density drops and the shear increases sharply. We note that q-profiles of this type have been measured recently for sawtoothing discharges in ASDEX [18] and TCA [19]. Such profiles allow a resistive mode to be turned on quickly, even at high $S$. The drop in the current profile outside the $q \approx 1$ region becomes increasingly steep at high $S$, and this partly compensates for the decrease in resistive growth rates with $S$. A second change connected with the equilibrium is that the variation of the central $q$ over the sawtooth cycle decreases with increasing $S$. Thus, the magnetic flux to be reconnected decreases, which has the effect of reducing the dependence of the crash time on $S$.

To summarize, reduced MHD simulations, in which the equilibria are self-consistently computed, give sawtooth periods, precursor growth rates and crash times that are in agreement with those of ohmically heated tokamaks for $S$ up to about $10^7$. This good agreement gives considerable, although admittedly indirect, evidence that sawteeth in small and medium size tokamaks are caused primarily by the resistive kink mode, similar to the mechanism originally proposed by Kadomtsev [2]. However, the q-profiles found in our study differ from the parabolic profile in Kadomtsev's analytic calculation by having a flat region with $q \approx 1$, outside which $q$ increases sharply, somewhat similar to the quasi-interchange model proposed by Wesson [5]. These properties are essential to obtain the rapid turn-on and crash observed experimentally.

Our simulations do not accurately reproduce sawteeth in large tokamaks, $S > 10^7$. This is not surprising, since our reduced MHD model neglects toroidal and finite-beta effects [28, 37]. From a theoretical point of view, it is clear that for high values of $S$, toroidal and finite pressure effects will influence the behaviour of the sawteeth. It appears probable that there is a transition between sawtooth governed by an essentially cylindrical resistive kink mode at low $S$ and sawtooth governed by toroidal, and possibly ideal, instabilities at higher $S$. On the basis of the present study, we estimate that the transition occurs for $S$ in the range of $10^7$ for ohmically heated tokamaks.

Experimental evidence for the assumption that ideal modes are involved in triggering of sawteeth in large tokamaks has come from JET [4, 5, 38]. It is also evident that the simplest reduced MHD model without any toroidal effects cannot reproduce sawteeth in which the central $q$ is below unity over the whole sawtooth cycle, as reported from TEXTOR [3].

Finally, since, generally speaking, the ideal MHD stability at $q = 1$ is close to marginal, many non-MHD effects may be able to significantly modify the behaviour of the sawteeth. Among these, we mention temperature and resistivity perturbations (Section 4.2), drift effects, the influence of hot particles expected in certain RF heating scenarios and, for future experiments, alpha particles [39]. In view of several unexpected results found in the present reduced MHD simulations and the variety of sawteeth observed under different experimental conditions, we are well aware that new surprises may be in store when self-consistent simulations are carried out, using more complete models at Lundquist numbers characteristic of large tokamaks.

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