Resistive toroidal stability of internal kink modes in circular and shaped tokamaks

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The linear resistive magnetohydrodynamical stability of the $n=1$ internal kink mode in tokamaks is studied numerically. The stabilizing influence of small aspect ratio [Holmes et al., Phys. Fluids B 1, 788 (1989)] is confirmed, but it is found that shaping of the cross section influences the internal kink mode significantly. For finite pressure and small resistivity, curvature effects at the $q=1$ surface make the stability sensitively dependent on shape, and ellipticity is destabilizing. Only a very restricted set of finite pressure equilibria is completely stable for $q_0 < 1$. A typical result is that the resistive kink mode is slowed down by toroidal effects to a weak resistive tearing/interchange mode. It is suggested that weak resistive instabilities are stabilized during the ramp phase of the sawtooth by effects not included in linear resistive magnetohydrodynamics. Possible mechanisms for triggering a sawtooth crash are discussed.

I. INTRODUCTION

The stability properties of the internal kink mode are of interest for understanding the sawtooth oscillations in tokamaks. Kadomtsev\(^1\) suggested that the sawteeth are triggered by an internal kink mode of toroidal mode number $n=1$, and that the nonlinear evolution of this instability leads to complete resistive reconnection. Bussac et al.\(^2,3\) studied the magnetohydrodynamical (MHD) stability of the internal kink by means of a large aspect ratio expansion. In contrast with the cylindrical result, they found that the ideal internal kink mode is stable in toroidal geometry when the poloidal beta at the $q=1$ surface is below a threshold value that depends on the current profile.\(^2\) Similarly, the resistive mode is slowed down from a resistive kink (ideally marginal with $\Delta' = \infty$) at an infinite aspect ratio to a weaker tearing instability ($0 < \Delta' < \infty$) at finite aspect ratio.\(^3\) Recent experimental investigations show that the safety factor in the center of tokamaks can be well below unity,\(^4,5\) even after the sawtooth crash,\(^6,7\) while other measurements have found $q$ profiles that are flat and close to unity in the central region.\(^8,9\) It has been shown computationally that the resistive internal kink can be linearly stable if the shear at the $q=1$ surface is sufficiently weak.\(^4,10\) This holds, in particular, when $q_0$ is well below unity, provided the shear is locally reduced at the $q=1$ surface, i.e., for a current profile with "shoulders" at $q=1$, as is found on the TEXTOR tokamak (tokamak experiment for technically oriented research).\(^4\) Theory also predicts that the internal kink mode can be stabilized by the presence of hot particles.\(^11,12\) The sawtooth oscillations have been stabilized experimentally in discharges with $q_0 < 1$, and the stabilization has been attributed either to current profile modification\(^13\) or to hot particle effects.\(^14\) It is clear that non-MHD effects can strongly modify the stability of the internal kink mode.\(^15-17\) Nevertheless, it is of interest to understand its stability properties within the simplest theoretical framework of resistive MHD. This is the goal of the present paper.

Even within magnetohydrodynamics, the internal kink mode is sensitive to a large number of parameters.\(^10,18\) Here, we examine the effects of aspect ratio, shaping of the cross section, current profiles, pressure, and wall separation. Growth rates of the $n=1$ mode are computed numerically from the full, resistive, compressible MHD equations for toroidal equilibria, using the recently developed, spectral stability code MARS\(^19\) and the equilibrium code CHEASE.\(^20\)

For very low pressure and circular cross section, our results confirm those of Holmes et al.\(^10\) Stability is favored by small aspect ratio, low shear at $q=1$, and low $q_0$. However, even slight pressure severely restricts the region of complete resistive MHD stability when $q_0 < 1$. This result can be understood analytically by considering the resistive and ideal interchange criteria, which play important roles for the linear stability of the internal kink. At finite pressure, resistive instability is hard to avoid because the resistive interchange criterion tends to be violated at $q=1$, as a result of noncircularity. A typical result, when pressure and shape effects are taken into account, is that the internal kink mode is slowed down by toroidal effects, and turns into a weak resistive tearing/interchange mode at small resistivity. Elliptic shaping renders pressure gradients considerably more destabilizing than for circular flux surfaces by violation of the resistive interchange criterion.
II. DEPENDENCE ON ASPECT RATIO, SHAPING, AND WALL SEPARATION

A. Specification of equilibria

We first study the effects of aspect ratio, shear at the $q=1$ surface and wall separation for different cross sections at zero pressure. The plasma-vacuum boundary of the equilibrium is prescribed as

$$ R = R_0 + a \cos(\theta + \delta \sin \theta), $$

$$ Z = a \kappa \sin \theta, $$

(1)

where $a$ is the minor radius, $R_0$ is the major radius, $\kappa$ is the elongation, and $\delta$ is the triangularity. For the zero-pressure study, we have chosen three different cross sections: circular ($\kappa=1, \delta=0$), elliptic ($\kappa=1.7, \delta=0$), and JET2' shape ($\kappa=1.7, \delta=0.3$).

The current profile is specified by the surface-averaged toroidal current density,

$$ I^* = \frac{\int J_\phi (J/R) d\chi}{\int (J/R) d\chi} $$

(2)

[where $J$ is the Jacobian for the transformation from flux coordinates $(\psi, \chi, \phi)$ to Cartesian coordinates] as a function of the normalized poloidal flux $\psi/\psi_{abs}$. Here $I^*(\psi/\psi_{abs})$ is prescribed, except for a multiplicative factor that is adjusted to specify the $q=1$ radius.

We first consider two current profiles that give a rather uniform and low shear inside a certain radius $\rho = \rho_p \approx 0.44$ (where $\rho(\psi) \equiv [V(\psi)/V_{pol}]^{1/2}$ is a normalized minor radius and $V(\psi)$ denotes the volume enclosed by a flux surface). For $\rho < \rho_p$, $I^*$ is a quadratic polynomial in $\psi^{1/2}$ with zero slope at $\rho = \rho_p$ but nonzero slope at the origin. Outside $\rho_p$, the current density falls rapidly and the shear rises. These current profiles will be referred to as flattened. In this section, we fix the $q=1$ radius at $\rho \approx 0.40$, i.e., in the low-shear region inside the “knee” of the current profile at $\rho = \rho_p$. Two different values of the central shear have been chosen, such that

$$ s = \frac{\rho dq}{q d\rho} $$

(3)

at $q=1$ is about 0.04 and 0.07, and $q_0 \approx 0.935$ and 0.88, respectively. The shear varies slightly with the aspect ratio and the shape. Figure 1 shows $I^*(\rho)$, $q(\rho)$, and $s(\rho)$ for a circular equilibrium with aspect ratio $A = R_0/a = 3$ and a flattened current profile with low central shear, $s_{q=1} = 0.04$.

The mass density and resistivity $\eta$ are taken constant in space. The resistivity is indicated by the Lundquist number $S = \tau_c / \tau_A$, where $\tau_c = \mu_0 a^2 / \eta$ is the resistive time connected with the minor radius and $\tau_A = R_0 / v_A$ is the toroidal Alfvén time. The growth rates are normalized with respect to $\tau_A$.

B. Fixed boundary results for zero pressure

At low pressure, the resistive internal kink mode is stabilized by small aspect ratio and weak shear at the $q=1$ surface. This is confirmed in Fig. 2, which shows the fixed boundary growth rates of the resistive kink mode versus $\varepsilon \equiv a/R_0$ at a Lundquist number of $S = 10^6$ for the flattened current profile. The various curves refer to the three cross sections (circle, ellipse, and JET shape) and the two different values of central shear. The resistive internal kink at zero pressure is stabilized for aspect ratios below a threshold that varies inversely with the shear.

Figure 2 shows that also shaping is important for internal kink stability, in particular, at large aspect ratio. For example, the elliptic case ($\kappa=1.7, \delta=0$) shows strong instability as the aspect ratio increases. The destabilization by ellipticity has been analyzed for ideal modes at large aspect ratio. It is connected to contributions proportional to $(\kappa-1)^2$ in the normalized potential energy $\delta W/a^2$. The destabilizing elliptic term competes with the...
FIG. 2. Fixed boundary resistive growth rates $\gamma$ at $S = 10^6$ versus inverse aspect ratio $\varepsilon$, for zero pressure equilibria with different shapes: circle, ellipse (squares), and JET shape (triangles). The flattened current profile has been used with different central shear (closed symbols: low shear, open symbols: high shear).

$O(\varepsilon^2)$ stabilizing toroidal contribution. At fixed shape, the elliptic shaping terms dominate over the toroidal term when the aspect ratio increases, and an elliptical equilibrium with $q_0 < 1$ is ideally unstable at large aspect ratio. As seen from Fig. 2, the resistive internal mode is significantly destabilized by an ellipticity of $\kappa = 1.7$ for aspect ratios of interest. However, a triangularity of $\delta = 0.3$, in combination with the same ellipticity (JET shape), improves stability over the circle. This is connected to stabilizing terms of order $\delta^2$ and $\varepsilon(\kappa - 1)\delta$ in $\delta W/a^2$, which become significant at large aspect ratio.

At low aspect ratio, the stabilizing toroidal effects tend to dominate over the shaping effects. Figure 2 shows that, for the low shear profile, the three shapes are stabilized at roughly the same aspect ratio, $A = R_0/a \approx 3$. For the higher central shear, the circular and JET-shaped equilibria are stable for $A$ less than about 1.5 and 1.8, respectively. The equilibria with higher central shear and an elliptic cross section are never completely stable, and the resistive growth rate increases again for $\varepsilon > 0.6$.

To summarize, the results of Fig. 2 for the fixed boundary internal kink at zero pressure, toroidicity and weak shear at the $q=1$ surface are stabilizing and ellipticity is destabilizing, but a combination of ellipticity and sufficient triangularity is more stable than a circular equilibrium.

C. Free boundary results for zero pressure

Next, we consider the effects of a free boundary. Figure 3(a) shows the growth rate of the $n=1$ mode at $S = 10^6$ for a circular zero beta equilibrium (the case of weak central shear in Fig. 2). One curve gives the result for a fixed boundary and the two others apply to free boundary modes with a conducting wall placed at a minor radius of $b = 1.2a$.

At large aspect ratio, the two free boundary modes correspond in an unambiguous way to their cylindrical counterparts: one is the internal "$m=1$" and the other is the external "$m=2$" mode. For the equilibria considered here, the external mode is stable with the wall on the plasma, but it becomes unstable for wall radii $b > 1.1a$. Figure 3(a) shows that, as the aspect ratio decreases, the "$m=1$" mode becomes increasingly stable, whereas the free boundary "$m=2$" mode is only weakly affected by...
toroidicity. At a certain aspect ratio ($A \approx 3$ for this case) the two branches cross over and the identification of internal or "$m=1$" and external or "$m=2$" breaks down. For aspect ratios below the crossover, the branch connected to the large aspect ratio internal mode acquires a dominant $m=2$ magnetic component and transforms into an "external" mode with a growth rate almost independent of $A$, and the large aspect ratio external branch is stabilized. (Note that the "$m=2$" mode is independent of aspect ratio only for low pressure. For higher pressure, the toroidal effects on the "$m=2$" mode are stabilizing because of favorable curvature at the $q=2$ surface.) Figures 3(b)-3(d) show the radial displacement and perturbed magnetic flux for the two different modes at aspect ratio $A=5$ and the single unstable mode at $A=2.5$. The mixture of $m=1$ and $m=2$ with displacements localized around the $q=1$ and $q=2$ surfaces is evident, and the relative sign of the $m=1$ and $m=2$ components is different for the two branches. For the more unstable branch, the $m=1$ and $m=2$ magnetic perturbations reinforce one another on the outboard side.

The cases shown in Fig. 3 indicate that the current profile must be stable to the $m=2/n=1$ tearing mode in the straight tokamak approximation, in order to be completely stable at finite aspect ratio and zero pressure. One way to stabilize the $m=2$ tearing mode is to decrease $q_0$ to values substantially below unity. However, for such profiles, the shear must be reduced locally at the $q=1$ surface in order for the $m=1$ resistive kink to remain stable. Thus, at zero pressure, free boundary stability can be achieved at finite aspect ratio by a TEXTOR profile with $q_0$ well below unity and shoulders in the current profile, which reduce the shear at $q=1$. Figure 4 shows an example of such a profile. The free boundary growth rates for this equilibrium are shown as functions of aspect ratio in Fig. 5. The external mode is now stable at all aspect ratios and the internal mode is stabilized for aspect ratios below approximately 10. Thus the TEXTOR profile with zero pressure is completely resistively stable for aspect ratios typical of tokamaks. The shear at $q=1$ for this equilibrium is $s \approx 0.035$, which is similar to the flattened profile of low central shear in Fig. 2, but the internal mode is stabilized at much larger aspect ratio for the TEXTOR profile.

The examples in Figs. 3-5 show that the stability of the resistive internal kink at zero pressure is sensitive to the current profile, aspect ratio, and wall position. Coupling to the external "$m=2$" mode becomes important at low aspect ratio. Stability to both internal and external free boundary modes at zero pressure requires nonmonotonic current profiles of the TEXTOR type. In the following, we shall mainly consider the purely internal modes by imposing a fixed boundary.

## III. PRESSURE EFFECTS

### A. Theory

A main factor for the stability of the internal kink is pressure. The global effects of pressure are described primarily by the poloidal beta at the $q=1$ surface. Through this paper, we use the definition

$$\beta_p(\psi) \equiv -\frac{4}{\mu_0 I_\phi^2(\psi) R_m} \int_0^\psi \frac{dp}{d\psi'} V(\psi') d\psi',$$

where $I_\phi(\psi)$ is the toroidal current flowing through a constant-$\psi$ surface, $V(\psi)$ is the enclosed volume, and $R_m$ is the major radius of the magnetic axis. The poloidal beta at the $q=1$ surface will be referred to as $\beta_p$. The large aspect ratio theory of Bussac et al. for circular cross sections predicts a stability limit in $\beta_p$ between 0 and 0.4 for the ideal internal kink. It should be noted that this limit is sensitive to the current profile.

In addition to global effects, pressure also has local effects on interchange stability. These are of particular im-
FIG. 5. Free-boundary resistive growth rates $\gamma$ for $S = 10^6$ versus inverse aspect ratio $\epsilon$ for a circular equilibrium with TEXTOR current profile and zero pressure. A conducting wall is assumed at $b=1.2a$. Note the absence of an \textquotedblleft$m=2$\textquotedblright branch and the complete stability at low aspect ratio.

importance for equilibria with low shear at $q=1$. We find that the combination of low shear and ellipticity easily leads to violation of the resistive, and even the ideal, interchange criterion. When the ideal interchange criterion is violated on the $q=1$ surface, a global \textquotedblleft$m=1$/n=1\textquotedblright mode becomes unstable, generally with a large growth rate.

The ideal\textsuperscript{24} and resistive\textsuperscript{25} interchange criteria read as

$$-D_I = \frac{1}{4} - \frac{1}{r} \left[ \frac{2}{2} \right]^2 - \frac{1}{r} \left( \frac{1}{2} - \frac{1}{2} \right) (g^2 I_1 + I_4) > 0,$$

$$-D_R = -D_I - \left( H - \frac{1}{2} \right)^2 > 0,$$

where $g = g(\psi) = RBq$ and

$$H = \frac{g}{q} \left( I_2 - I_5 \right) \frac{g^2 I_1 + I_4}{g^2 I_1 + I_4}.$$

The effects of shaping can be seen in the simplified expressions obtained from a large aspect ratio expansion. The flux surfaces are assumed to have the form

$$R = R_0 - \left[ r - E(r) \right] \cos(\omega - \Delta(r)) + T(r) \cos 2\omega + R_0 O(\epsilon^2),$$

$$Z = \left[ r + E(r) \right] \sin(\omega + T(r)) \sin 2\omega + R_0 O(\epsilon^3),$$

where $E$ is the elliptic deformation (related to the elongation by $\kappa = 1 + 2E/r + O((E/r)^2)$), $\Delta(r)$ is the Shafranov shift, and $T(r)$ is the triangular deformation [related to the triangularity by $\delta = 4T(r)/r$]. Keeping the terms proportional to $E$ and $T$ to first order, we obtain from an expansion to second order in $\epsilon$ [where $E/r$ is considered as $O(\epsilon^0)$ and $T/r$ as $O(\epsilon^1)$] and $\mu_0$ is set to 1:

$$-D_I = 1 + \frac{2p'}{r B^2} \frac{q^2}{2} \left[ 1 - q^2 + \frac{3q^2}{4} \left( \frac{E}{r} + E' \right) + \frac{3}{2} \Lambda \left( \frac{E}{r} - E' \right) \right] - \frac{R_0 q^2}{r} \left[ \frac{2ET}{r^2} + \frac{7ET^'}{2r - \frac{2}{2} ET^'} \right],$$

$$-D_R = 1 + \frac{2p'}{r B^2} \frac{q^2}{2} \left[ 1 - q^2 + \frac{3q^2}{4} \left( \frac{E}{r} + E' \right) + \frac{3}{2} \Lambda \left( \frac{E}{r} - E' \right) \right] - \frac{R_0 q^2}{r} \left[ \frac{2ET}{r^2} + \frac{7ET^'}{2r - \frac{2}{2} ET^'} \right] + rqq' \left[ -\Lambda + \left( \frac{3\Lambda}{2} + \frac{1}{4} \right) E' \right] + \frac{R_0 (ET + ET^' + \frac{3}{2} ET^')}{r}.$$
FIG. 6. Stability results for a circular equilibrium with a flattened current profile, low central shear, and $\beta_p = 0.05$. (a) Resistive growth rates $\gamma$ for $S = 10^7$ versus inverse aspect ratio $\varepsilon$. The fixed boundary results are shown as closed circles and the open circles refer to a conducting wall at radius $b = 1.2a$. (b) The resistive interchange parameter $-D_R$ (closed circles) and ellipticity $\varepsilon$ (open circles) at the $q=1$ surface.

FIG. 7. Stability results for a weakly oblate equilibrium with flattened current profile, low central shear, and $\beta_p = 0.05$. (a) Free boundary resistive growth rates $\gamma$ for $S = 10^7$ versus inverse aspect ratio $\varepsilon$ with a conducting wall at radius $b = 1.2a$. The open circles show the growth rate and the closed circles show the real part of the frequency. (b) The resistive interchange parameter $-D_R$ (closed circles), ellipticity $\varepsilon$ (open circles), and shear $s$ (open triangles) at the $q=1$ surface.

cross sections: circle: $\kappa = 1$, $\delta = 0$ (Fig. 6); weakly oblate: $\kappa = 0.9$, $\delta = 0$ (Fig. 7); and JET shape: $\kappa = 1.7$, $\delta = 0.3$ (Fig. 8). In all cases, $\beta_p = 0.05$ and $S = 10^7$, and we have used the flattened current profile with low central shear and $q_0 \approx 0.935$. The pressure profiles are characterized by $dp/d\psi = \text{const}$, except for the outer 10% of the poloidal flux, where pressure gradient goes smoothly to zero at the edge.

Figure 6(a) shows the resistive growth rates versus $\varepsilon$ for equilibria with a circular boundary, $\beta_p = 0.05$, and two different wall positions $b = a$ and $b = 1.2a$. By comparison with Fig. 2, the major effect of finite pressure is that the fixed boundary mode remains unstable also at low aspect ratio. This mode is now dominated by the $m=1$ component. The mode is weakly dependent on the wall position and is driven unstable by interchange effects. Figure 6(b) shows that the resistive interchange criterion becomes increasingly violated at low aspect ratio. The principal reason for this appears to be the small natural ellipticity of the internal flux surfaces at finite aspect ratio. The ellipticity $\varepsilon_{q=1}$ of the $q=1$ surface is shown in Fig. 6(b). With a circular boundary, the ellipticity of the internal surfaces is, to leading order, proportional to $\varepsilon^2 [\varepsilon_{q=1} \approx 5.98 \times 10^{-2} (a/R_0)^2$ for the equilibria in Fig. 6]. For $A < 3$, the destabilizing ellipticity correction in the resistive interchange criterion dominates over the shear term. Thus, even though the plasma boundary is circular, the $O(\varepsilon^2)$ modifications of the shape of the internal surfaces change the stability of the internal kink significantly at relevant aspect ratios. This current profile is particularly sensitive to "small" effects because of the low shear on $q=1$, but similar behavior is observed for the equilibrium with higher central shear in Sec. II B. [For the sequence of equilibria in Fig. 6, where the current profile is held fixed, the shear depends weakly on the aspect ratio, but this is not significant. It should be remarked that the large aspect ratio expansion that led to (6) is not strictly valid when the ellipticity is of order $\varepsilon^2$, but (6) nevertheless seems to give a good approximation.]

Figure 7(a) shows the growth rate for weakly oblate equilibria with $\kappa = 0.9$ and $\beta_p = 0.05$. With decreasing aspect ratio, these equilibria first become overstable, and are then stabilized. The resistive interchange criterion is satisfied, because the $q=1$ surface remains oblate also for small aspect ratios [$\varepsilon_{q=1} \approx (-4.09 + 6.6a^2/R_0^2) \times 10^{-2}$]. For these oblate equilibria, the internal kink mode is completely stable at low aspect ratio and moderate pressure. Even though both the deviation from circular boundary ($\kappa = 0.9$) and the $S$ number ($10^7$) are modest, the resistive internal mode behaves quite differently than in the case of a circular boundary. With more pronounced shaping and larger $S$ the influence of curvature of course becomes stronger.
IV. CURRENT PROFILE EFFECTS

A. Circular shape

In this section, we study the effects of the current profile in combination with finite pressure and shaping and discuss how a sawtooth crash might be triggered by changes in the current profile. We first consider circular cross section and the two types of current profiles used in Sec. II: the flattened (I) and the TEXTOR profile (II). To generate families of self-similar profiles, we apply a uniform scaling of the toroidal current density, keeping the poloidal beta and the internal inductance fixed (see Ref. 20). The scaling approximately corresponds to multiplying the q profile by a constant and allows to prescribe the q value at a specified radius, \( q_p = q_p^0 \) or the q=1 radius.

For the flattened profile (I), the shear is low in a central region and has a local minimum at \( \rho = \rho_p = 0.44 \), where \( dI*/d\rho = 0 \). Outside this radius, the shear increases rapidly. The minimum shear \( s(\rho_p) \) is about 0.03 and the central safety factor \( q_c \) is given by \( q_c \approx 0.935q_p \). For \( q_p > 1 \), the q=1 surface is in the central region of low shear, but when \( q_p \) is decreased below unity, it moves into the outer region of rapidly increasing shear. Minimum shear at \( q = 1 \) occurs for \( q_p = 1 \). A sequence of self-similar equilibria with decreasing \( q_p \) may correspond approximately to the time evolution during the ramp phase of a sawtooth, if the preceding crash leads to complete reconnection and almost flat central \( q_c \) followed by peaking of the current due to trapped particle effects. The pressure profile \( \frac{p(\psi)}{p_0} \) is the same as in Sec. III B. The central beta is related to \( \beta_p \) at \( q = 1 \) by \( \beta_0 \approx 0.096\beta_p \) and the volume average beta is \( \langle \beta \rangle = 2\mu_0\langle p \rangle / \langle B^2 \rangle \approx 0.039\beta_p \).

Figures 9 and 10 show the growth rates of the internal kink mode for the flattened current profile as functions of \( q_p \) at different pressures and resistivities. The aspect ratio is 4 and a conducting wall is assumed at radius \( b = 1.2a \). Complete resistive, free-boundary stability is never achieved for the flattened current profile. However, for \( \beta_p < 0.05 \), the resistive growth rate is small when the \( q = 1 \) surface is in the region of small shear, and the mode is
FIG. 10. Free-boundary growth rates versus $q_p$. All parameters are identical to Fig. 9, except $S = 6 \times 10^6$.

predominantly external with a large $m=2$ magnetic perturbation. Such weak instabilities may well be stabilized by effects not included in linear resistive MHD.

By comparing Fig. 9 for $S = 6 \times 10^6$ and Fig. 10 for $S = 6 \times 10^3$, one can identify regions of resistive and ideal instability. The instabilities for $\beta_p > 0.15$ and $q_p < 1$ (when the $q=1$ surface is in the outer region of high shear) are ideal. The normalized growth rates are several times $10^{-3}$ and are almost independent of the resistivity. The growth rate peaks when the shear at $q=1$ is small, as expected for ideal modes. By contrast, for $\beta_p < 0.05$, the instabilities are resistive. The growth rates follow the tearing scaling with respect to resistivity and have a minimum when the shear is small at $q=1$. Figure 10 shows that the resistive growth rates are very small at high $S$. These growth rates do not even come close to those observed experimentally in sawtooth precursors where, typically, $\gamma/\omega_A > 10^{-3}$.

An interesting feature is evident for the cases with $\beta_p > 0.15$. The pressure-driven instabilities are sensitive to the value of $q_p$, or to the location of the $q=1$ surface with respect to the knee of the current profile. The finite-beta growth rates have maxima when the $q=1$ surface is at the radius of minimum shear and remain high when the $q=1$ surface reaches the outer, high-shear region. Thus, a “pressure-driven” instability may be triggered by changes in the current profile rather than by an increase in the pressure itself. If we invoke stabilization of a weak resistive-MHD instability by some unspecified mechanism, a sawtooth crash could be triggered by the increase in MHD growth rates when the $q=1$ surface approaches, or moves out into, the outer high-shear region. Such a trigger mechanism for the sawteeth may explain why the inversion radius remains almost constant between successive crashes. A knee in the current profile created by reconnection during the preceding crash would only be partially smoothed out by resistive diffusion before the next crash, and could then act as a spatially localized magnetic trigger for the following sawtooth.

For comparison with other geometries, we note that the resistive interchange criterion generally indicates stability for circular cross section with $A=4$, but $-D_R$ takes small positive values (due to finite shear rather than favorable curvature). The growth rates are generally slightly larger for $\beta_p = 0$ than for $\beta_p = 0.05$ because of the increased inertia associated with the motion along the field lines in the finite beta case.

The details of the results in Figs. 9 and 10 depend on the current profile. For instance, if the central shear is reduced, the ideal pressure-driven instabilities are enhanced, while the growth rates of the resistive instabilities for low pressure are reduced.

Next, we consider the TEXTOR current profile. The shoulders in $I^*$ have been adjusted so that the shear has a minimum of about 0.034 at $\rho = \rho_1 \approx 0.44$, the central $q$ is $q_0 \approx 0.634 q_p$, and the aspect ratio is 4. Figures 11 and 12 show the growth rates for different $S$ and central pressures. The behavior is similar to that for the centrally flat profiles, but the TEXTOR profile supports about twice the pressure before becoming ideally unstable. At high pressure, $\beta_p > 0.2$, the growth rates are very sensitive to the position of the $q=1$ surface. For the TEXTOR profile, there is indeed an interval in $q_p$, where the equilibrium is entirely stable. However, this interval is small, and certainly less than the

FIG. 11. Free-boundary growth rates for $S = 6 \times 10^6$ versus the safety factor $q_p$ at the radius of minimum shear ($\rho = 0.44$) for a circular equilibrium with TEXTOR current profile and different values of $\beta_p$. The aspect ratio is 4 and the conducting wall is at $b = 1.2a$.

FIG. 12. Free-boundary growth rates versus $q_p$. All parameters are identical to Fig. 11, except $S = 6 \times 10^3$. 

Figure 13. Free-boundary growth rates versus the safety factor \( q_p \) at the radius of minimum shear \((\rho = 0.44)\) for TEXTOR equilibrium. All parameters are identical to Fig. 12, except the aspect ratio \( A = 2.5 \). (a) shows the full range of \( \gamma \) and (b) is a blowup to show the stable region.

shift in \( q \) during the sawtooth cycle. This again indicates that the resistive kink mode is stabilized during most of the sawtooth cycle by effects not included in linear resistive MHD.

As discussed in Sec. II, low aspect ratio is stabilizing for the internal kink mode. An example is given in Fig. 13 which shows the growth rate \( \gamma \) as a function of \( q_p \) for a sequence of equilibria with aspect ratio \( A = 2.5 \) and a TEXTOR current profile. Figure 13 refers to \( S = 6 \times 10^8 \) and differs from Fig. 12 only with respect to the aspect ratio. We note that the region of complete stability is larger at the smaller aspect ratio and that there is even a small interval in \( q_p \) giving complete resistive stability for \( \beta_p = 0.20 \).

**B. JET shape**

As noted in Sec. II, the ellipticity of the \( q=1 \) surface in a JET-shaped cross section can cause ideal instability at moderate pressure if the shear at \( q=1 \) is small. In order not to be dominated by interchange effects, we consider current profiles with higher shear in this section. To show the dependence on the shear, we choose two values of the minimum shear for each of the two types of profiles: flattened, denoted (I-H) and (I-L) for high and low shear, respectively, and TEXTOR, denoted (II-H) and (II-L).

The shear at \( q=1 \) is shown in Fig. 14 for the four profiles versus \( q_p = q(\rho = \rho_p \approx 0.41) \). The fixed boundary growth rates at \( S = 6 \times 10^8 \) are shown for different pressures in Figs. 15-17.

Figure 15 shows that at zero pressure the growth rates are similar for the four different current profiles. The growth rates at zero pressure are generally lower than for the circular case at \( A = 4 \). However, the JET cross section is more sensitive to pressure, and a clear increase in growth rates resulting from \( \beta_p = 0.05 \) is evident in Fig. 16. None of the current profiles is resistively stable for this pressure in JET geometry. However, the large shear profiles have small growth rates for \( \beta_p = 0.05 \), in particular, the TEXTOR profile.
TOR current profile. The pressure-driven instability for the low shear TEXTOR profile is highly sensitive to the $q$ value.

For higher pressure, $\beta_p = 0.10$ (Fig. 17), the two centrally flat profiles both give rather large growth rates for all values of $q_p$, whereas the TEXTOR profile gives normalized growth rates as small as a few times $10^{-4}$ when $q_p > 1$, i.e., when the $q=1$ surface is inside the shoulder.

C. Oblate cross section

To illustrate the importance of the average curvature, we again consider the slightly academic example of an oblate cross section (elongation $\kappa = 0.9$, zero triangularity, and aspect ratio $A = 4$). The current profile is of the TEXTOR type with a minimum shear of 0.042. The growth rates for $\beta_p = 0.05$ and $S = 6 \times 10^7$ are shown in Fig. 18. When the $q=1$ surface is located near the radius of minimum shear, the growth rate becomes complex, and in a certain interval, $0.995 < q_p < 1.005$, the mode is stabilized, evidently as a result of favorable average curvature. The resistive interchange parameter, $-\Delta_B$, indicates stability and reaches a maximum of about 0.07 for $q_p = 1$.

For higher pressure, $\beta_p = 0.10$, and the oblate cross section, the destabilizing global effects of pressure dominate over the stabilizing layer effects, and the resistive internal kink is no longer stable for any $q_0 < 1$.

V. SUMMARY AND DISCUSSION

The internal kink mode is sensitive to a large number of effects, and the following is an attempt to delineate the most important of these.

For zero pressure, the resistive MHD stability of the internal kink is influenced primarily by the aspect ratio and the current profile $I^*(\psi)$. The low aspect ratio is stabilizing. Stability is improved by low shear at the $q=1$ surface, but also by low $q_0$. Equilibria with monotonic $I^*$ are stable to fixed boundary modes when the aspect ratio is below a
threshold value that varies inversely with the shear. Free-boundary stability appears to require nonmonotonic $I^*$. Current profiles of the TEXTOR type with shoulders near the $q=1$ surface are much more stable than monotonic profiles, and can remain resistively stable with a free boundary at very large aspect ratios.

The internal kink stability is affected by shaping already at zero pressure. The shaping effects are more pronounced at large aspect ratio, where the toroidal stabilization is weak. Ellipticity alone is destabilizing, but a combination of ellipticity and sufficient triangularity, such as the JET shape, is more stable than a circular equilibrium.

Central pressure gradients are generally strongly destabilizing for the resistive internal kink mode. Part of the reason for this is global (i.e., $\Delta'$) effects on the eigenfunction. However, interchange effects at the $\text{q}=1$ resonant surface are important, in particular, when the shear is low, and this makes the stability at finite pressure highly sensitive to shaping. For many shapes of interest, notably, the JET shape, the curvature at $\text{q}=1$ is unfavorable because of ellipticity, and the resistive interchange criterion is generally violated for low shear. For a JET-shaped cross section, we do not find any profile that is resistively stable with $q_0 < 1$ and $\beta_p > 0.05$. At low shear, even the Mercier criterion can be violated at pressures well below the Bussac limit. This typically leads to global ideal internal kinks with rather high growth rates. A more detailed study of the ideal stability will be presented elsewhere.

The resistive MHD stability is highly sensitive to curvature. For instance, even for equilibria with a circular boundary, the natural ellipticity of the $\text{q}=1$ surface can destabilize an internal kink by resistive interchange if the shear is low. For an oblate boundary, giving favorable curvature and resistive interchange stability at $\text{q}=1$, a finite window of free-boundary stability exists for moderate $\beta_p$.

A clear conclusion of the present study is that complete resistive MHD stability is difficult to achieve for $q_0 < 1$ and finite pressure. It appears unlikely that such stringent stability conditions can be met during the entire ramp phase of the sawtooth cycle when the internal kink is manifestly stable. When confronted with the experimental observations of $q_0 < 1$, this leads us to conclude that the linear resistive MHD model is overly pessimistic and that weak resistive instabilities are stabilized by effects not included in the model.

Nevertheless, finite aspect ratio effects are strongly stabilizing and often reduce the resistive kink to a weak resistive tearing/interchange mode with a small growth rate at high $S$. The growth rate can easily be well below $10^{-5}\Omega_A$, which is a characteristic growth rate of sawtooth precursors, e.g., in JET. If we adopt $10^{-5}\Omega_A$ (or a value of the order of the diamagnetic frequency) as an ad hoc threshold for stability, the stability boundary in large tokamaks will be close to the ideal stability boundary. (For such modes one may expect a significant increase in growth rate over the resistive MHD value from finite electron inertia.)

Several effects are known that can stabilize weak resistive instabilities. One such effect is nonlinear saturation at a finite island size. A finite island reduces the destabilizing layer effects that occur for elliptic shaping. The process of nonlinear saturation by the reduced destabilization from unfavorable curvature is the inverse of the nonlinear destabilization by finite island size in regions of favorable curvature, as discussed by Kotschenreuther et al. We mention that nonlinear simulations of the sawtooth activity based on the “straight tokamak” resistive-MHD model showed generally good agreement with experimental results for low Lundquist numbers ($S < 10^5$), while difficulties appeared for higher values of $S$, where more detailed physics effects should be important.

Other mechanisms for stabilizing weak resistive-MHD instabilities are offered by more detailed physics models, including diamagnetic rotation and trapped particle effects. Clearly, non-MHD effects, and/or nonlinearity, have to be taken into account in order to understand the behavior of the sawteeth in large tokamaks. The present study has, nevertheless, shown a number of parametric dependencies, and points out the importance of the curvature effects at the $q=1$ surface that become dominant at low shear. It appears that curvature effects should also be important when more detailed physics models are applied.

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\begin{thebibliography}{99}
\bibitem{Soltiwhish} H. Soltiwhish (private communication).
\end{thebibliography}


24C. Mercier, Nucl. Fusion 1, 47 (1960).


