Seminar
Introduction to Fuzzy Logic I

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**Figure:** Crisp

**Figure:** Fuzzy
Figure: Washing machine

Figure: Rice cooker
**Figure:** Unmanned helicopter

**Figure:** Tensiometer
**Figure:** Inverted pendulum
Example

A system with two continuous variables $x, y \in [0, 1]$, described by 2 rules,

$r_1$: If $x$ is big, then $y$ is small

$r_2$: If $x$ is very small, then $y$ is very big

What is the output for the input $x=0.4$?
Esquema
**How to represent predicates?**

$\leq_{P}$ is perceptive relation

$L$-degrees $\mu_{P_i} : X \rightarrow L$ ($i = 1, 2, 3$), where $L$ is an ordered structure.
Types of predicates:

- Rigid: consists in two classes. For instance, “To be even”
- Semirigid: consists in a finite number of classes
- Imprecise: consists in infinite classes. For instance, tall.
How to represent predicates? III

**Rigid ≡ Crisp**

\[ P = \text{“To be 8”} \]
Semirigid $\equiv$ finite number of classes
How to represent predicates? V

Each predicate $P$, could be represented by many fuzzy sets depending on its use.

“Philosophical investigations”, Wittgenstein (1953): The meaning of a predicate is its use in Language.

Different models of $P = \text{small in } X = [0, 10]$, with $L = [0, 1]$ depending on the use of $P$. 

![Graphs illustrating different models of smallness](image)
How to represent predicates? VI

Each use of the predicate is represented by a curve (by a fuzzy set).

\[ \mu : X \rightarrow [0, 1] \]

A [0, 1]-degree \( \mu_P \) is any function \( X \rightarrow [0, 1] \), such that

If \( x \leq_P y \), then \( \mu_P(x) \leq \mu_P(y) \),

It the inverse order introduce by the predicate \( \leq_P^{-1} \) is defined as

\[ x \leq_P^{-1} y \iff y \leq_P x. \]

So, \( x =_P y \) iff \( x \leq_P \) and \( x \leq_P^{-1} \).

If \( \mu_P(x) = r \), it is said that \( x \in_r P \).
Design the representation of the predicate: Context, purpose, use,...

**Example**

- Predicate `tall` in two different contexts, in a *school* or in a *basketball team*.

![Graphs showing the membership functions μ_{A_{col}} and μ_{A_{bal}} for the predicate tall.](image)
Family resemblance in $[0,1]^X$, $fr \subset \mathcal{F}^*(X) \times \mathcal{F}^*(X)$
(read $(\mu, \sigma) \in fr$: $\mu$ and $\sigma$ show family resemblance)

$$(\mu, \sigma) \in fr \iff \begin{cases} 
1) Z(\mu) \cap Z(\sigma) \neq \emptyset, S(\mu) \cap S(\sigma) \neq \emptyset \\
2) \mu \text{ is non-decreasing in } A \subset X \iff \sigma \text{ is non-decreasing in } A \\
3) \mu \text{ is decreasing in } A \subset X \iff \sigma \text{ is decreasing in } A 
\end{cases}$$

$S(\mu) = \{ x \in X; \mu(x) = 1 \}$ and $Z(\mu) = \{ x \in X; \mu(x) = 0 \}$. 
How to represent predicates? IX

**Example 1**

\[ Z(\mu) \cap Z(\sigma) = [0, 2] \]
\[ S(\mu) \cap S(\sigma) = \{10\} \Rightarrow (\mu, \sigma) \in \text{fr} \]

**Example 2**

\[ S(\mu) \cap S(\sigma) = \emptyset \Rightarrow (\mu, \sigma) \notin \text{fr} \]
**Opposite or antonym**

<table>
<thead>
<tr>
<th>P</th>
<th>aP</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>tall</td>
<td>short</td>
</tr>
<tr>
<td>small</td>
<td>big</td>
</tr>
</tbody>
</table>
Concerning the opposite $aP$ of $P$, this opposition is translated by

$$\leq_{aP} = \leq_{P}^{-1}$$

Hence, $\leq_{a(aP)} = (\leq_{P}^{-1})^{-1} = \leq_{P}$, that forces $a(aP) = P$.

For example, with $P = \text{tall}$, it is $aP = \text{short}$ and $a(aP) = \text{tall}$.

This property of $aP$ shows a way for obtaining $\mu_{aP}$ once $\mu_{P}$ is known. Let it $A : X \to X$ be a symmetry on $X$, that is a function such that

- If $x \leq_{P} y$, then $A(y) \leq_{P} A(x)$
- $A \circ A = \text{id}_{X}$
Once $\mu_P : X \rightarrow L$ is known, take $\mu_{aP}(x) = \mu_P(A(x))$, for all $x$ in $X$, that is, $\mu_{aP} = \mu_P \circ A$. Function $\mu_{aP} = \mu_P \circ A$ is a degree for $aP$, since:

- $x \leq_{aP} y \iff y \leq_P x \Rightarrow A(x) \leq_P A(y) \Rightarrow \mu_P(A(x)) \leq \mu_P(A(y))$, and verifies,

- $\mu_{a(aP)} = \mu_{(aP)} \circ A = (\mu_P \circ A) \circ A = \mu_P \circ (A \circ A) = \mu_P \circ id_X = \mu_P$. 

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Opposite and negate IV

Example
Opposite of small

A?
Example 2

Opposite of around 4

\[ A(x) = 10 - x \]
**Example 2**

Opposite of around 4

\[ A(x) = \begin{cases} 
4 - x, & \text{if } x \leq 4 \\
14 - x, & \text{if otherwise}
\end{cases} \]
Let it $P$ be a predicate in $X$, and $P' = \neg P$ its negate. Concerning the negation $P'$ of $P$, this negation is translated by

$$\leq_{P'} \subset \leq_{P}^{-1}$$

Since,

If $x$ is less $P$ than $y$, then $y$ is less $\neg P$ than $x$, or, equivalently, $\leq_{P} \subset \leq_{P'}^{-1}$. 
Let it \( N : L \to L \) be a function such that

1. If \( a \leq b \), then \( N(b) \leq N(a) \), for all \( a, b \) in \( L \)

2. \( N(\alpha) = \omega \), and \( N(\omega) = \alpha \), being \( \alpha \) the infimum and \( \omega \) the supremum of \( L \).

with such a function \( N \), it is

\[
\mu_{P'} = N \circ \mu_P
\]

an \( L \)-degree for \( P' \), since

\[
x \leq_{P'} y \Rightarrow y \leq_P x \Rightarrow \mu_P(y) \leq \mu_P(x) \Rightarrow \\
N(\mu_P(x)) \leq N(\mu_P(y)) \iff \mu_{P'}(x) \leq \mu_{P'}(y).
\]
Provided the negation function does verify

3. \( N \circ N = \text{id}_L \),

then

\[
\mu_{(P')}'(x) = N(\mu_P'(x)) = N(N(\mu_P(x))) = (N \circ N)(\mu_P(x)) = \text{id}_L(\mu_P(x)) = \mu_P
\]

for all \( x \) in \( X \), or \( \mu_{(P')}' = \mu_P \).

Functions \( N \) verifying (1), (2), and (3) are called strong negations.
If $L = [0, 1]$, there is a family of strong negations widely used in fuzzy set theory, the so-called Sugeno’s negations:

$$N_\lambda(a) = \frac{1 - a}{1 + \lambda a}, \text{ with } \lambda > -1, \text{ for all } a \in [0, 1].$$

For example, $N(a) = 1 - a$, $N_1(a) = \frac{1-a}{1+a}$, $N_{-0.5} = \frac{1-a}{1-0.5a}$, $N_2(a) = \frac{1-a}{1+2a}$, etc.

Since obviously,

$$N_{\lambda_1} \leq N_{\lambda_2} \iff \lambda_2 \leq \lambda_1,$$

it results:

- If $\lambda \in (-1, 0]$, then $N_0 \leq N_\lambda$
- If $\lambda \in (0, +\infty]$, then $N_\lambda < N_0$. 
Opposite and negate XI

Graphically,
With $N_0(a) = 1 - a$, and $A(x) = 10 - x$ in $X = [0, 10]$, if

$$
\mu_{\text{big}}(x) = \begin{cases} 
0, & \text{if } x \in [0, 4] \\
\frac{x-4}{4}, & \text{if } x \in [4, 8] \\
1, & \text{if } x \in [8, 10],
\end{cases}
$$

results

$$
\mu_{\text{small}}(x) = \mu_{\text{big}}(10 - x) = \begin{cases} 
0, & \text{if } x \in [6, 10] \\
\frac{6-x}{4}, & \text{if } x \in [2, 6] \\
1, & \text{if } x \in [0, 2],
\end{cases}
$$

$$
\mu_{\text{not big}}(x) = 1 - \mu_{\text{big}}(x) = \begin{cases} 
1, & \text{if } x \in [0, 4] \\
\frac{8-x}{4}, & \text{if } x \in [4, 8] \\
0, & \text{if } x \in [8, 10],
\end{cases}
$$

whose graphics
show that the pair \((big, small)\) is coherent, since \(\mu_{small} \leq \mu_{not\ big}\).

- Notice that \(\mu_{aP} \leq \mu_{P'}\)
Remarks

- Notice that $\mu_a P \leq \mu P'$
- $a(aP) = P$, but not always $(P')' = P$. (empty/full)
- Notice the main difference of a symmetry and a negation $A : X \rightarrow X$ and $N : L \rightarrow L$
Let $P$, $Q$ be two predicates in $X$.
Consider the new predicates ‘$P$ and $Q$’, and ‘$P$ or $Q$’, used by means of:

- ‘$x$ is $P$ and $Q$’ $\Leftrightarrow$ ‘$x$ is $P$’ and ‘$x$ is $Q$’
- ‘$x$ is $P$ or $Q$’ $\Leftrightarrow$ Not (Not ‘$x$ is $P$’ and Not ‘$x$ is $Q$’). (Duality)
**INTERSECTION**

Take L-degrees $\mu_P, \mu_Q$.

Given $(L, \leq)$, let $* : L \times L \rightarrow L$ be an operation, verifying the properties

- $a \leq b, c \leq d \Rightarrow a * c \leq b * d$
- $a * c \leq a, a * c \leq c$,

Then, $\mu_P(x) * \mu_Q(x)$ is an L-degree for $P$ and $Q$ in $X$, since:

$x \leq_P x \leq_Q y \Rightarrow x \leq_P y$ and $x \leq_Q y \Rightarrow \mu_P(x) \leq \mu_P(y)$ and $\mu_Q(x) \leq \mu_Q(y) \Rightarrow \mu_P(x) * \mu_Q(x) \leq \mu_P(y) * \mu_Q(y)$. 
Hence,

\[ \mu_P(x) \ast \mu_Q(x) = \mu_{P \text{ and } Q}(x), \]

is an \textbf{L-degree} for ‘\( P \text{ and } Q \)’ in \( X \), and the operation \( \ast \) can be called an \textit{and-operation}.

Notice that it is,

\[ \mu_{P \text{ and } Q}(x) \leq \mu_P(x), \text{ and } \mu_{P \text{ and } Q}(x) \leq \mu_Q(x). \]
If $\ast$ is an and-operation, and $N : L \rightarrow L$ is a strong negation, define

$$a \oplus b = N(N(a) \ast N(b)), \text{ for all } a, b \in L.$$ 

- Since $a \leq b, c \leq d \Rightarrow N(b) \leq N(a), N(d) \leq N(c) \Rightarrow N(b) \ast N(d) \leq N(a) \ast N(c) \Rightarrow N(N(a) \ast N(c)) \leq N(N(b) \ast N(d))$, it results $a \oplus c \leq b \oplus d$.

- Analogously, from $N(a) \ast N(b) \leq N(a)$, it follows $a \leq N(N(a) \ast N(b)) = a \oplus b$, and $b \leq a \oplus b$, for all $a, b$. 
Then, \( \mu_P(x) \oplus \mu_Q(x) \) is an L-degree for \( P \) or \( Q \).

\[
x \leq_P \text{ or } Q \ y \iff x \leq (P' \text{ and } Q') \ y \iff
\]

\[
y \leq_{P'} \text{ and } Q' \ x \Rightarrow \mu_{P'} \text{ and } Q'(y) \leq \mu_{P'} \text{ and } Q'(x) \iff
\]

\[
\mu_{P'}(y) * \mu_{Q'}(y) \leq \mu_{P'}(x) * \mu_{Q'}(x) \Rightarrow
\]

\[
N(\mu_P(y)) * N(\mu_Q(y)) \leq N(\mu_P(x)) * N(\mu_Q(x)) \Rightarrow
\]

\[
N(N(\mu_P(x)) * N(\mu_Q(x)) \leq N(N(\mu_P(y)) * N(\mu_Q(y)) \iff
\]

\[
\mu_P(x) \oplus \mu_Q(x) \leq \mu_P(y) \oplus \mu_Q(y).
\]

Hence, \( \mu_P(x) \oplus \mu_Q(x) \) can be taken as an L-degree for ‘\( P \) or \( Q \)’ in \( X \), and the operation \( \oplus \) can be called an or-operation.
Remark

- The existence of operations $\ast$ and $\bigcirc$ in $L$ warrants the existence of $L$-degrees for and, or, respectively.
In fact, other basic properties of $\ast$ and $\oplus$ are collected in the definition of a Basic Flexible Algebra (BFA).

A BFA is a seven-tuple $\mathcal{L} = (L, \leq, 0, 1; \ast, \oplus, ')$, where $L$ is a non-empty set, and

1. $(L, \leq)$ is a poset with minimum 0, and maximum 1.
2. $\ast$ and $\oplus$ are mappings (binary operations) $L \times L \to L$, such that:
   1. $a \ast 1 = 1 \ast a = a$, $a \ast 0 = 0 \ast a = 0$, for all $a \in L$
   2. $a \oplus 1 = 1 \oplus a = 1$, $a \oplus 0 = 0 \oplus a = a$, for all $a \in L$
   3. If $a \leq b$, then $a \ast c \leq b \ast c$, $c \ast a \leq c \ast b$, for all $a, b, c \in L$
   4. If $a \leq b$, then $a \oplus c \leq b \oplus c$, $c \oplus a \leq c \oplus b$, for all $a, b, c \in L$
3. $' : L \to L$ verifies
   1. $0' = 1$, $1' = 0$
   2. If $a \leq b$, then $b' \leq a'$
4. It exists $L_0, \{0, 1\} \subset L_0 \varsubsetneq L$, such that with the restriction of the order and the three operations $\ast, \oplus$, and $'$ of $\mathcal{L}$, $\mathcal{L}_0 = (L_0, \leq, 0, 1; \ast, \oplus, ')$ is a boolean algebra.
A continuous t-norm is with mappings $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ verifying the following properties,

- Continuity in both variables
- Associativity $T(T(x, y), z) = T(x, T(y, z))$, $\forall x, y, z \in [0, 1]$
- Commutativity $T(x, y) = T(y, x)$ $\forall x \in [0, 1]$
- Monotonicity $T(x, y) \leq T(z, t)$ if $x \leq z, y \leq t$
- $T(x, 1) = x$, $\forall x \in [0, 1]$
- $T(x, 0) = 0$, $\forall x \in [0, 1]$
Union and intersection IX

⊕ FOR FUZZY SETS (T-CONORMS)

A continuous t-conorm is with mappings $S : [0, 1] \times [0, 1] \to [0, 1]$ verifying the following properties,

- **Continuity in both variables**
- **Associativity** $S(S(x, y), z) = S(x, S(y, z))$, $\forall x, y, z \in [0, 1]$
- **Commutativity** $S(x, y) = S(y, x)$ $\forall x \in [0, 1]$
- **Monotonicity** $S(x, y) \leq S(z, t)$ if $x \leq z$, $y \leq t$
- $S(x, 0) = x$, $\forall x \in [0, 1]$
- $S(x, 1) = 1$, $\forall x \in [0, 1]$
Given $P$, if $MP = \text{Not } P \text{ and Not } aP$, with $N$ for not, $A$ for the opposite, and $\ast$ for and, results

$$\mu_{MP}(x) = \mu_{P'} \text{ and } (aP)'(x) = \mu_{P'}(x) \ast \mu_{(aP)'}(x) = N(\mu_P(x)) \ast N(\mu_P(A(x))),$$

for all $x$ in $X$. 
Linguistic modifiers or linguistic hedges, $m$, are adverbs acting on $P$ just in the concatenated form $mP$. For example, with $m = \text{very}$ and $P = \text{tall}$, it is $mP = \text{very tall}$.

Among linguistic modifiers there are two specially interesting types:

- **Expansive modifiers**, verifying $\text{id}_{\mu_P(x)} \leq \mu_m$,
- **Contractive modifiers**, verifying $\mu_m \leq \text{id}_{\mu_P(x)}$. 
With the expansive, it results
\[
\text{id}_{\mu_P(x)}(\mu_P(x)) = \mu_P(x) \leq \mu_m(\mu_P(x)) = \mu_{mP}(x) : \mu_P(x) \leq \mu_{mP}(x), \text{ for all } x \text{ in } X.
\]

With the contractive, it results
\[
\mu_{mP}(x) = \mu_m(\mu_P(x)) \leq \text{id}_{\mu_P(x)}(\mu_P(x)) = \mu_P(x) : \mu_{mP}(x) \leq \mu_P(x), \text{ for all } x \text{ in } X.
\]

In \( L = [0, 1] \) with the Zadeh’s old definitions,
\[
\mu_{\text{more or less}}(a) = \sqrt{a}, \quad \mu_{\text{very}}(a) = a^2.
\]

Which one is contractive?
A linguistic variable \textbf{LV} is formed after considering

1. Its principal predicate, \( P \)
2. One of the opposites of \( P \), \( aP \)
3. Some linguistic modifiers \( m_1, \ldots, m_n \),
   and by adding:
4. Its negate (\textit{not} \( P \)), or the middle-predicate (\textit{MP}), or \( P \ and \ Q \), or \textit{not} \( m_1P \), or \( P \ and \ m_2aP \), ...
Then LV, is called the linguistic variable generated by $P$, and reflects the linguistic granulation perceived for the concept. For example,

- $LV =$ Age, is Age $\{ \text{young, old, middle-aged, not old, not very young,..} \}$
- $LV =$ Temperature, is Temp $\{ \text{cold, hot, warm, not cold, not very hot,..} \}$
- $LV =$ Size, is Size $\{ \text{large, small, medium, very large,..} \}$
Linguistic Variable III

Usually in fuzzy control are used only three labels, for instance:

![Graph showing fuzzy sets for cold, warm, and hot temperatures]
Extension Principle

× : \( \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), can be extended to \( \otimes : [0, 1]^{\mathbb{R}} \times [0, 1]^{\mathbb{R}} \rightarrow [0, 1]^{\mathbb{R}} \), by means of Extension Principle

\[
(\mu \otimes \sigma)(t) = \sup_{t = x \times y} \min(\mu(x), \sigma(y)).
\]
Example extending the operation of $+$

Let’s calculate the sum for the following fuzzy sets:

- $\mu = 0/0 + 1/1 + 0/3 + 0/4$
- $\sigma = 0/0 + 0/1 + 0.5/2 + 1/3 + 0/4$

$$\mu \oplus \sigma = 0/0 + 0/1 + 0/2 + 0.5/3 + 1/4 + 0.5/5 + 0/6$$

Since,

- $(\mu \otimes \sigma)(0) = \sup_{0=x\times y} \min(\mu(x), \sigma(y)) = \sup(\min(\mu(0), \sigma(0)))$.
- $(\mu \otimes \sigma)(1) = \sup_{1=x\times y} \min(\mu(x), \sigma(y)) = 
  \sup(\min(\mu(1), \sigma(0)), \min(\mu(0), \sigma(1))) = \sup(0, 0) = 0.$
- $(\mu \otimes \sigma)(2) = \sup_{2=x\times y} \min(\mu(x), \sigma(y)) = 
  \sup(\min(\mu(2), \sigma(0)), \min(\mu(0), \sigma(2)), \min(\mu(1), \sigma(1))) = 
  \sup(0, 0, 0) = 0.$
Extension Principle III

- \((\mu \otimes \sigma)(3) = \sup_{3=xy} \min(\mu(x), \sigma(y)) = \sup(0, 0, 0.5, 0) = 0.5.\)

- \((\mu \otimes \sigma)(4) = \sup_{3=xy} \min(\mu(x), \sigma(y)) = \sup(0, 0, 1, 0.5, 0) = 1.\)

- ...
Extension Principle IV

\[ \mu_3 \quad \mu_3 \oplus \mu_3 \]

\[ 0 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \quad 1 \]
Thanks for your attention!